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PSYCHOMETRIC MONOGRAPHS

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— A STUDY OF REDUCED RANK MODELS FOR MULTIPLE PREDICTION —

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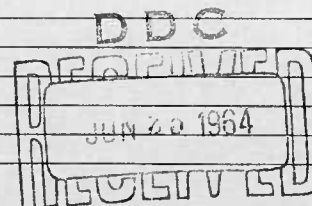
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A STUDY OF REDUCED RANK MODELS FOR MULTIPLE PREDICTION

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BY GEORGE R. BURKET

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A STUDY OF
REDUCED RANK MODELS FOR
MULTIPLE PREDICTION

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REDUCED RANK MODELS
FOR
MULTIPLE PREDICTION**

By

GEORGE R. BURKET

AMERICAN INSTITUTE FOR RESEARCH
AND
UNIVERSITY OF PITTSBURGH

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PREFACE

Prediction problems frequently arise in which the regression weights must be based on a relatively small number of criterion observations. In such cases, current techniques permit the utilization of only a very few predictors, even though many more may be available. Unless one or more of the predictors is closely related to the criterion, accurate predictions cannot be made. The possibility of increasing the accuracy of prediction under such circumstances through the use of reduced-rank methods is investigated in this study.

On the basis of normal regression theory, a general reduced-rank model is formulated in terms of prediction from factor scores. The problems of selecting a method of factoring, of selecting an optimal subset of prespecified size from among a given set of factors, and of selecting an optimal rank are considered. It is shown that in the absence of criterion observations, the optimally chosen reduced-rank solution will be the one that accounts for the greatest proportion of variance in the full-rank predictor matrix. Prediction either from subsets of the original predictors, which are equivalent to triangular factors, or from principal-axes factors is considered. It is concluded that, when degrees of freedom are sufficiently limited, the most accurate predictions obtainable will be those based on the largest principal-axes factors. As a tentative solution to the problem of optimal rank, estimates are derived which are intended to indicate the accuracy of prediction to be expected when regression weights computed on the basis of data in one sample are applied to data in other samples.

An empirical comparison of five reduced-rank methods is carried out, employing a variety of ranks, sample sizes, and criteria. The five methods include prediction from the principal-axes factors, selected in three different ways, and from the original predictors, selected in two different ways. The results indicate that weights computed by the method of largest principal-axes factors on samples with as few as 30 cases can give predictions as accurate as those from weights computed by conventional techniques on samples of several hundred cases.

The present monograph was submitted as a doctoral dissertation at the University of Washington in July 1962. The writer wishes to thank his sponsor, Professor Paul Horst, for the invaluable blend of criticism and encouragement that he provided. The work for the present monograph was largely supported by Office of Naval Research Contract Nour. 477(33) and Public Health Research Grant M-743(C7) (principal investigator: Paul Horst). Acknowledgment is due Mrs. Judy Goodstein and Mrs. Helen Ranek for their work in typing and proofreading the manuscript.

GEORGE R. BURKET

Pittsburgh, Pennsylvania
October, 1963

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CHAPTER 1

INTRODUCTION

Basic Requirements

Accurate predictions of an individual's degree of success or failure in such socially significant activities as a college course, training for some vocation, or a particular job would be of incalculable utility, both to the individual concerned and to the community. Remarkably accurate predictions of this nature can be obtained with existing statistical techniques, provided that two basic requirements are satisfied. First, there must be measurements available on a number of variables related to performance in the activity of interest. It must be possible to obtain these measurements on any individual before he engages in the activity. Second, such measurements must be obtained for a large number of persons who subsequently engage in the activity.

The first requirement can almost always be met. Indeed, it is usually possible to find many variables having at least some relation to performance in the criterion activity. To obtain measurements on a large number of variables may be expensive, but accurate predictions of many activities are of sufficient value to warrant large expenditures. The second requirement is much less likely to be satisfied, since the number of persons who actually engage in a particular activity is often limited. This is particularly true for activities requiring an unusual degree of ability, where accurate predictions are apt to be most desired. Many socially significant activities are full-time occupations which individuals must pursue for years before their success or failure can be determined. If the number of persons engaging in such an activity is too small to permit application of existing techniques, no feasible expenditure will yield accurate predictions. We need new techniques.

The Statistical Model

A system for obtaining the best possible predictions for a given criterion would be the following. First, determine all variables, termed predictors, not statistically independent of the criterion. Then obtain measurements of predictors and criterion on a sufficiently large validation sample so that every possible configuration of predictor values is represented by a large number of cases. Compute the criterion mean for each of these configurations. To make a prediction for a particular case, determine the configuration of the predictors for that case. The prediction will be the criterion mean for cases in the validation sample having that configuration.

Such a system is unworkable because of practical limitations on sample size and number of predictors. Under certain circumstances, moreover, a much simpler system could give equally accurate predictions. If, for example, the criterion means were known to be functionally related to the predictors, it would only be necessary to determine this function. In practice, such a functional relation is virtually always assumed. It may also happen that a small subset of all variables statistically related to the criterion will give predictions as accurate as the entire set. Even where a very large number of independent predictors is readily available, the number that may actually be used is limited by the available sample size. This is because it is necessary to have many more cases than there are parameters in the assumed functional relation between predictors and criterion mean. Otherwise one could not obtain stable estimates of these parameters.

In least-squares or regression theory and also in correlation theory, the mean of the criterion is assumed to be a linear function of the predictors. In correlation theory, predictors and criterion are assumed to be random variables having a joint multivariate normal distribution. In regression theory, the criterion is assumed to be a normally distributed random variable, while the predictors are thought of as being fixed. Anderson (1958, p. 61) recommends using one model or the other depending on whether or not the predictors may be considered random. Mood (1950, p. 312) states that, in practice, most correlation problems can be more appropriately handled by regression methods. In many cases, the two models have led to equivalent procedures; under the null hypothesis, estimates of regression weights, test criteria, and probability theory are all the same. However, when the null hypothesis (*viz.*, that predictors and criterion are independent) is not true, the probability theory differs.

In prediction problems in psychology, the predictor variables are generally random rather than fixed, and the null hypothesis is rarely true. Thus correlation theory would appear to be more appropriate. However, since correlation theory is considerably more complex and difficult to apply than regression theory, the latter is generally used, with the hope that the practical differences between conclusions drawn from the two models will be negligible. In the present study, prediction problems will for the most part be considered within the context of regression theory.

It may prove useful at this point to make the distinction between actual prediction problems and validation problems. In validation problems, the goal is to demonstrate a systematic relationship between a number of "independent variables" and a "dependent variable." To accomplish this, one formulates the null hypothesis of no relationship and hopes to reject it at some level of confidence. Thus, for validation problems, correlation theory and regression theory are equivalent. In prediction problems, on the other hand, the null hypothesis is assumed to be false. The goal is to obtain a

regression equation which, when applied to predictor measures in future samples, will give the most accurate estimate possible of the corresponding criterion values. Having obtained such a regression equation, one would also wish to have estimates or confidence intervals indicating the accuracy to be expected when the regression equation is applied to new samples. In validation problems, the multiple correlation is often used as a measure of relationship between the dependent and independent variables. It is sometimes termed a validity coefficient, or simply a validity. In prediction problems, the correlation between the prediction and the criterion in new samples may be used as a measure of accuracy of prediction. Such a coefficient may be termed a weight-validity to distinguish it from the multiple correlation coefficient between the prediction battery and the criterion in the original sample.

Purpose of the Study

The present study is concerned with prediction problems as opposed to validation problems. Regression theory in its current form is adequate for those applications in which the available number of cases far exceeds the available number of predictors, i.e., in which the number of degrees of freedom is large. In such cases, weight-validity will be very close to battery validity, and the least-squares estimates of the regression weights will provide optimal predictions. But when the number of predictors available is relatively large in relation to sample size, as is perhaps more often than not the case, problems arise that lack satisfactory theoretical answers. One such problem is that of estimating an index, such as weight-validity, that will provide some idea of the accuracy of prediction to be expected in new samples. A more important problem is that of determining the regression weights which will give the most accurate predictions possible in new samples.

These optimal weights will not in general be given by the conventional least-squares solution applied to all available predictors. For example, if the number of predictors is the same as the number of cases in the sample, the least-squares weights for an arbitrary subset of predictors will usually give better weight-validity (though lower validity) than the weights for the entire set. More generally, in such an extreme case, any lower-rank approximation to the matrix of predictor values would give better predictions than the complete matrix. As the situation becomes less and less extreme, there must come a point where some ranks and some methods of rank reduction and not others are preferable to the complete matrix. At a still less extreme point, the entire set of predictors will presumably give better predictions than any reduced-rank approximation. Still, when predictors are discarded, the loss of accuracy of prediction may be so slight as to be more than offset by the practical savings of not having to measure as many predictors.

Thus in any prediction problem where the number of degrees of freedom

is limited, the question of rank reduction arises: can the complete predictor matrix be improved upon, and if so, which method of reduction and which rank will give the greatest improvement? When its purpose is to give more accurate prediction by increasing degrees of freedom, the much-studied predictor selection problem is a special case of the rank-reduction problem. Predictor selection methods are more often used, however, in situations where an upper limit on the size of the prediction battery is given by considerations of cost. The emphasis is thus on obtaining an optimal set of predictors of a particular size rather than on obtaining optimal predictions regardless of battery size. Perhaps because of the prevalence of the former emphasis, particularly before the advent of electronic computers, the problem of predictor selection has received a great deal more attention than the general problem of rank reduction.

Most methods of predictor selection are alike in selecting first the variable having the highest single validity, and adding, step by step, the variable which, together with those previously selected, will give the greatest increase in the multiple correlation with the criterion. These so-called accretion methods differ with respect to computational procedure and method of deciding how many predictors to use. Perhaps the computationally simplest such method is the square-root (or triangular-factorizing) method described by Summerfield and Lubin (1951). Horst has generalized and extended this method for absolute (1955) and differential (1954) prediction of multiple criteria. Horst and MacEwan (1960) have described a method which is essentially the reverse of the accretion method. Here one eliminates at each step the predictor contributing least to the multiple correlation. The accretion and elimination methods will not in general result in the same battery, nor will either of them necessarily give the battery of given size having the highest obtainable validity.

Horst (1941) has suggested two models for reduced-rank prediction. His rationale is based upon the factor analysis hypothesis that the predictor matrix is basic only because of the presence of error or specific factors. One of these models assumes the presence of specifics. Accordingly, the matrix of predictor intercorrelations is augmented by the vector of criterion correlations and communality estimates are placed in the diagonal prior to factoring. Least-squares weights are then computed for the common factors. This method was tested by Leiman (1951) using 12 predictors and computing weights on samples of 30 cases. A rank-3 solution gave weight-validities which were significantly higher than those obtained with the full-rank solution. This method has the disadvantage of being difficult to treat theoretically, since the nature of communalities and of the factor scores (which are not unique) are not well understood. The other model suggested by Horst accomplishes rank reduction by attempting to remove error factors rather than specific factors. Here the best least-squares approximation to the predictor intercorrelation matrix is

used, the principal-axes solution. One advantage of this method is that it is theoretically straightforward. Another advantage is that rank reduction is accomplished independently of the criterion and thus does not capitalize on the errors in the criterion.

Virtually the exact opposite of this model has been implicitly suggested by Guttman (1958). Since the inverse of the predictor correlation matrix is directly involved in computing regression weights, one might well base predictions on the best lower-rank approximation to the inverse rather than on the approximation to the intercorrelation matrix. The best set of factors for approximating the inverse is, as Guttman points out, the worst for approximating the intercorrelation matrix. In view of this paradox, perhaps one should abandon approximation as a criterion for selecting the factors to be retained for prediction and simply use those factors giving the highest multiple correlation, as is attempted in the predictor-selection methods. Certainly the basic assumption of the rationale for approximating the intercorrelation matrix may be questioned: that the reliable variance is concentrated in the larger principal-axes factors, the smaller factors being composed mainly of error. For example, in a study by Davis (1945) involving nine principal-axes factors, a strict correspondence between variance contribution and reliability was not found; e.g., the split-half reliability for the eighth factor was larger than for the fourth factor.

The present study proceeds along both theoretical and empirical lines. First an attempt is made to work out some of the consequences of regression theory for reduced-rank models. Since, as noted above, there is reason to question the appropriateness of regression theory for psychological prediction problems, an empirical comparison of five reduced-rank procedures is also carried out. The methods used were predictor elimination, predictor selection, the method of approximating the intercorrelation matrix, the method of approximating the inverse, and the method using the principal-axes factors giving the highest multiple correlation. As will be seen, both the theoretical and the empirical evidence favors the method of approximating the intercorrelation matrix.

CHAPTER 2

IMPLICATIONS OF REGRESSION THEORY FOR REDUCED RANK MODELS

The General Linear Hypothesis

Regression theory was first worked out at the beginning of the 19th century by Gauss and Legendre and has since, of course, been presented by innumerable authors from various points of view. Among recent sources, a rigorous presentation with geometrical interpretations has been given by Scheffé (1959). A simpler presentation entirely in terms of matrix algebra is given by Kempthorne (1952). Anderson (1958) provides a generalization to multiple criteria. A presentation in terms of deviation scores may be found in Cramér (1946). Some results from regression theory which are relevant to the rank-reduction problem are summarized below. The derivations, which are for the most part omitted, may be found in the sources mentioned above. Let

- y be a column vector of N observations on the criterion;
- x be an $N \times M$ matrix of rank $M < N$, each row of which represents an observation on each of M predictors;
- e be an N th-order column vector of uncorrelated errors, each distributed normally with mean zero and variance σ^2 ;
- β be an $M \times 1$ vector of population regression coefficients;
- C be a covariance matrix of the variable given in the subscript.

The general linear hypothesis is that

$$(1) \quad y = x\beta + e.$$

The assumptions regarding e , apart from normality, may be stated as

$$(2) \quad E(e) = 0,$$

$$(3) \quad C_e = E(ee') = \sigma^2 I.$$

From these equations it follows that the criterion has the expectation

$$(4) \quad E(y) = x\beta,$$

and the covariance matrix

$$(5) \quad C_y = E[(y - x\beta)(y - x\beta)'] = \sigma^2 I.$$

Let

$\hat{\beta}$ be the $M \times 1$ vector of least-squares estimates of the population regression coefficients;

\tilde{y} be the $N \times 1$ vector of estimates of the criterion based on $\hat{\beta}$.

Then

$$(6) \quad \hat{\beta} = (x'x)^{-1}x'y,$$

and

$$(7) \quad \tilde{y} = x\hat{\beta}.$$

The vector $\hat{\beta}$ has the property of minimizing the sum of squares of the errors in estimating y from \tilde{y} . These errors will be orthogonal to the predictors and also to the estimates themselves. The error sum of squares has the expectation

$$(8) \quad E[(y - \tilde{y})'(y - \tilde{y})] = (N - M)\sigma^2.$$

Thus

$$(9) \quad \hat{\sigma}^2 = \frac{(y - \tilde{y})'(y - \tilde{y})}{N - M}$$

provides an unbiased estimate of σ^2 . What is generally termed the standard error of estimate is given by $\hat{\sigma}$. The variable $\hat{\sigma}^2$ is distributed independently of $\hat{\beta}$.

The estimates of the regression coefficients have the expectation

$$(10) \quad E(\hat{\beta}) = \beta,$$

and the covariance matrix

$$(11) \quad C_{\hat{\beta}} = E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] = \sigma^2(x'x)^{-1}.$$

The estimates of the criterion have the same expectation as the criterion itself,

$$(12) \quad E(\tilde{y}) = E(x\hat{\beta}) = xE(\hat{\beta}) = x\beta,$$

but are not independent, since from (7), (11), and (12),

$$(13) \quad C_{\tilde{y}} = E[(x\hat{\beta} - x\beta)(x\hat{\beta} - x\beta)'] = xC_{\hat{\beta}}x' = \sigma^2x(x'x)^{-1}x'.$$

The canonical form of the general linear hypothesis may be obtained as follows. Let x be expressed as

$$(14) \quad x = ub',$$

where u is an $N \times M$ orthonormal matrix of factor scores, and b is an $M \times M$ matrix of factor loadings. Let V be an N by $N - M$ orthonormal matrix such that the $N \times N$ matrix H in

$$(15) \quad H = [u \quad v]$$

is an orthonormal matrix. The matrices u , b , and v are always obtainable, and can be determined solely from the predictors without reference to the criterion. Then the N th-order vector of transformed criterion values

$$(16) \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = H'y = \begin{bmatrix} u'y \\ v'y \end{bmatrix}$$

has the expectation

$$(17) \quad E(z) = \begin{bmatrix} E(z_1) \\ E(z_2) \end{bmatrix} = \begin{bmatrix} b'\beta \\ 0 \end{bmatrix},$$

and the covariance matrix

$$(18) \quad C_z = \sigma^2 I.$$

Thus the best possible predictions for the $N - M$ transformed observations z_2 will always be zero, regardless of the true regression coefficients or of the particular values of the criterion. The least-squares estimates of the regression weights are so chosen as to reproduce exactly the M transformed observations z_1 from

$$(19) \quad z_1 = u'y = b'\hat{\beta},$$

so that

$$(20) \quad \hat{\beta} = b'^{-1}u'y.$$

Equation (20) may also be obtained by putting (14) in (6). Thus, errors can occur only in estimating z_2 , and since the estimated value of z_2 is zero, we have

$$(21) \quad (y - \hat{y})'(y - \hat{y}) = z_2'z_2.$$

Metric and the Status of the Multiple Correlation

In regression theory, the multiple correlation coefficient and other functions of the predictors such as means, standard deviations, and covariances do not have the status of population parameters. This is because the predictors are not assumed to be random variables but rather fixed values. Thus, regression theory does not admit of statistical inferences about such functions. However, one can make statistical inferences about such characteristics of future samples as depend on the criterion, provided that the relevant features of the predictor matrix in the future samples are assumed to be known in advance. For example, one can assume that exactly the same predictor matrix will be obtained in future samples or merely that the predictor intercorrelations will be the same. Using the latter assumption and scaling the criterion appropriately, one can define both a sample and a population multiple correlation coefficient.

Except where correlations are concerned, no assumptions about metric are made in the present paper. However, it should be noted that if the equations of the preceding section were to be applied to data in the original units of observation, a correction for origin would be required. This correction will be accomplished if a predictor is added which is defined to be unity for all cases. If this is done, equation (6) of the preceding section may be shown to be identical to the usual formulas for raw-score regression weights, which are typically expressed in terms of means and covariances or correlations and standard deviations.

The question of metric also arises in connection with defining multiple correlation. The assumption made here whenever correlation coefficients are discussed is that all measures are normalized, i.e., expressed as deviations from the sample mean in units of the sample standard deviation multiplied by the square root of the number of cases in the sample. We may now define the square of the multiple correlation in the sample as

$$(22) \quad R^2 = \hat{\beta}'x'x\hat{\beta} = y'x(x'x)^{-1}x'y$$

and in the population as

$$(23) \quad \rho^2 = \beta'x'x\beta.$$

If we let r be the $M \times M$ matrix of predictor intercorrelations, (23) may be written as

$$(24) \quad \rho^2 = \beta'r\beta,$$

since, on the basis of the assumption about the metric,

$$(25) \quad r = x'x.$$

Thus ρ will be a population parameter if it is assumed that the predictor intercorrelations will be the same in all samples.

An unbiased estimate for ρ may be obtained as follows. The expectation of the criterion sum of squares is, from (1),

$$(26) \quad E(y'y) = E[(x\beta + e)'(x\beta + e)] = \beta'x'x\beta + 2\beta'x'E(e) + E(e'e).$$

From (23), the first term on the right is ρ^2 and from (2) the second term is zero. The third term is the trace of (3). Thus

$$(27) \quad E(y'y) = \rho^2 + N\sigma^2.$$

Since the errors of estimate are orthogonal to the estimates, we have

$$(28) \quad y'y = \tilde{y}'\tilde{y} + (y - \tilde{y})'(y - \tilde{y}).$$

From (7) and (22), the first term on the right is R^2 . Thus from (8) and (27),

$$(29) \quad \begin{aligned} E(R^2) &= E(y'y) - E[(y - \tilde{y})'(y - \tilde{y})] \\ &= \rho^2 + N\sigma^2 - (N - M)\sigma^2 = \rho^2 + M\sigma^2. \end{aligned}$$

Given the assumed metric, the criterion sum of squares will always be unity, so from (27),

$$(30) \quad \sigma^2 = \frac{1 - \rho^2}{N}$$

and (29) may be written as

$$(31) \quad E(R^2) = \rho^2 + \frac{M(1 - \rho^2)}{N}.$$

From (31) it is clear that the extent to which R^2 overestimates ρ^2 will vary directly with the number of predictors and inversely with the sample size. Solving equation (31) for ρ^2 we obtain the following unbiased estimate for ρ^2 :

$$(32) \quad R_c^2 = \frac{NR^2 - M}{N - M}.$$

Equation (32) will be recognized as the familiar "shrinkage" formula for multiple R .

It is perhaps worth noting that R_c , or "shrunk R " is not an estimate of weight-validity or of the shrinkage to be expected in the correlation between the criterion and its estimate if weights computed on one sample are applied in other samples. It does provide an estimate of the correlation that would have been obtained between the criterion and its estimate if the population regression weights had been used instead of their least-squares estimates. Shrunk R may also be thought of as an estimate of the multiple R that could be obtained in a very large sample having the same predictor intercorrelation matrix as the observed sample.

The Accuracy of Prediction in Future Samples

In prediction problems we wish to compute a set of weights from a given sample which will give the most accurate predictions obtainable when applied to other samples. Specifically, we will assume that the sum of squares of the errors of prediction in each other sample is the quantity to be minimized. If we let $\bar{\beta}$ be a set of weights obtained in some fashion from a previous sample, this sum of squares may be written (Kempthorne, 1952) as

$$(33) \quad (y - x\bar{\beta})'(y - x\bar{\beta}) = (y - x\hat{\beta})'(y - x\hat{\beta}) \\ + e'x(x'x)^{-1}x'e + 2(\beta - \bar{\beta})'x'e + (\beta - \bar{\beta})'x'x(\beta - \bar{\beta}).$$

The expected value is

$$(34) \quad E[(y - x\bar{\beta})'(y - x\bar{\beta})] = N\sigma^2 + (\beta - \bar{\beta})'x'x(\beta - \bar{\beta}).$$

Now the second term on the right has an expectation in the sample from

which $\bar{\beta}$ was obtained. Assuming that the usual least-squares estimates are employed, we have, using equation (11),

$$(35) \quad E[(\beta - \hat{\beta})'x'x(\beta - \hat{\beta})] = \text{tr} [E[x(\beta - \hat{\beta})(\beta - \hat{\beta})'x']] \\ = \text{tr} (xC_{\beta}x') = \sigma^2 \text{tr} [x(x'x)^{-1}x'].$$

Using (14), we may write the matrix whose trace we require as

$$(36) \quad x(x'x)^{-1}x' = ub'(bb')^{-1}bu' = ub'b'^{-1}b^{-1}bu' = uu'.$$

Putting (36) in (35), we may write

$$(37) \quad E[(\beta - \hat{\beta})'x'x(\beta - \hat{\beta})] = \sigma^2 \text{tr} (uu') = \sigma^2 \text{tr} (u'u) = \sigma^2 \text{tr} (I) = M\sigma^2.$$

Now if we assume that $x'x$, or equivalently the factor-loading-matrix b , is the same in all samples, we would expect the sum of squares of errors of prediction to be $(N + M)\sigma^2$. More generally, if $\bar{\beta}$ is any estimate of β computed from the original sample, we would expect the sum of squares of errors of prediction in future samples, provided that the factor-loading matrix is the same as in the original sample, to be

$$(38) \quad \psi_{\bar{\beta}} = N\sigma^2 + E[(\beta - \bar{\beta})'x'x(\beta - \bar{\beta})].$$

Thus $\psi_{\bar{\beta}}$ will be taken as an inverse index of weight-efficiency: the smaller it is, the more suitable $\bar{\beta}$ will be for a prediction problem. In particular,

$$(39) \quad \psi_{\beta} = (N + M)\sigma^2.$$

Since the interpretation of (38) is basic to the following development, we will examine its derivation with some care. Certainly $\psi_{\bar{\beta}}$ is not a mathematical expectation in the usual sense, but rather an expectation of an expectation. Since N , σ^2 , β , and (by assumption) $x'x$ are fixed, the expectation in (34) is a function of $\bar{\beta}$, and is thus fixed as soon as the original sample is drawn. Since this quantity is a function of the criterion in the original sample, its expectation in this sample is $\psi_{\bar{\beta}}$. The quantity that we are directly concerned with minimizing is the one in (34). This quantity is itself not determined in advance of drawing the first sample, but its expectation is determined. Rather than minimize the quantity of direct interest, then, we attempt to minimize its expectation.

An estimate of weight-validity may be obtained from (39). Assuming the metric of the previous section, and using (9) and (22),

$$(40) \quad \sigma^2 = \frac{y'y - \hat{y}'\hat{y}}{N - M} = \frac{1 - R^2}{N - M}.$$

Thus, an unbiased estimate for ψ_{β} is, from (39)

$$(41) \quad \hat{\psi}_{\beta} = \frac{N + M}{N - M} (1 - R^2).$$

For an arbitrary set of weights $\bar{\beta}$, the weight-validity is

$$(42) \quad W = \frac{y'x\bar{\beta}}{\sqrt{\bar{\beta}'x'x\bar{\beta}}}.$$

The sum of squares of errors of prediction is

$$(43) \quad S = (y - x\bar{\beta})'(y - x\bar{\beta}) = 1 - 2y'x\bar{\beta} + \bar{\beta}'x'x\bar{\beta}.$$

If (42) is substituted in (43),

$$(44) \quad S = 1 - 2W\sqrt{\bar{\beta}'x'x\bar{\beta}} + \bar{\beta}'x'x\bar{\beta}.$$

Since $\bar{\beta}$ is the vector of least-squares weights from the original sample, under the assumption that $x'x$ is constant, the radical in the second term on the right of (44), and the third term on the right become, respectively, R and R^2 of the original sample. Solving (44) for W gives

$$(45) \quad W = \frac{1 + R^2 - S}{2R}.$$

Now to obtain an estimate of W , we substitute for S in (45) the estimate of its expectation given by (41). Simplifying, we obtain

$$(46) \quad \hat{W} = \frac{NR^2 - M}{R(N - M)}.$$

To see the relation of the estimated weight-validity to the estimated population multiple correlation as defined in the preceding section, we put (32) in (46), obtaining

$$(47) \quad \hat{W} = \frac{R_c^2}{R} = \frac{R_c}{R} R_c.$$

Since R_c is less than R (unless R is unity), the left-hand factor on the right of (47) will be less than one, so \hat{W} will be less than R_c .

Perhaps a more important application of (38) is its use as a criterion for evaluating reduced-rank models for computing regression weights. An alternate approach is indirectly suggested by Leiman (1951, pp. 3-4). There, the assumption is made that the least-squares weights for the lower-rank system will give better predictions than least-squares weights for the full-rank system to the extent that they provide closer approximations to the population regression weights for the full-rank battery. The reason for rejecting this position is as follows: It is well known that the optimal weights for a subset of predictors may differ greatly from the weights of the same predictors when the full battery is retained. A mathematical statement of this fact is given in (104). Thus one cannot properly measure the suitability of a reduced-rank set of weights in terms of how closely they approximate the full-rank weights. It seems more likely that the least-squares weights for

a subset of predictors or of factor scores may, because of the increased number of degrees of freedom, be so much more stable than the weights for the full set as to give more accurate predictions despite the loss of information. In any case, the criterion in (38) involves no assumptions other than those usually made in applications of regression theory to prediction problems and is, moreover, referred directly to the expected errors of prediction.

In evaluating reduced-rank solutions, a question arises as to the number of factors to be included in the general linear hypothesis. If the full-rank hypothesis is retained, then the quantity $N\sigma^2$ in (38) is fixed, so that the only way of improving on $\hat{\beta}$ will be to find a $\tilde{\beta}$ for which the second term is less than $M\sigma^2$. If, however, a smaller set of, say, L predictors (either the original ones or factor scores) is hypothesized, both terms change. The variance of the errors, σ^2 , will of course increase in proportion to the systematic variance in the criterion accounted for by the discarded predictors. If we denote this larger variance by σ_L^2 and the least-squares weights for the reduced battery by $\tilde{\beta}$, then

$$(48) \quad \psi_{\tilde{\beta}} = (N + L)\sigma_L^2,$$

as will be seen in the next section. Thus the $\tilde{\beta}$ for any subset of L predictors for which $(N + L)\sigma_L^2$ is less than $(N + M)\sigma^2$ will be an improvement over $\hat{\beta}$.

Another possible application of (38) would be in obtaining a criterion for how many predictors to retain in the standard predictor-selection procedures. If we denote by R_L the multiple correlation obtained with a set of L predictors, this criterion is obtained directly from (41):

$$(49) \quad \hat{\psi}_{\tilde{\beta}} = \frac{N + L}{N - L} (1 - R_L^2).$$

One would retain those L predictors for which $\hat{\psi}_{\tilde{\beta}}$ is the smallest. We use $\hat{\psi}_{\tilde{\beta}}$ rather than \hat{W} since weight-validity is an indication not of the actual errors of prediction but of the errors which would have been obtained if the predictions could themselves have been weighted after the criterion had been observed. In other words, a correlation coefficient between two variables is independent of differences in location and scale, whereas actual errors of prediction are in part determined by such differences.

The General Reduced-Rank Model

The reduced-rank solution will first be developed in terms of a general factor model. Predictor selection and prediction from principal-axes factors will then be considered as special cases of this model. Let

$$(50) \quad x'x = bb'$$

be any complete factoring of $x'x$. Then

$$(51) \quad u = x(b')^{-1}$$

will be the orthonormal matrix of factor scores. The matrices x , u , and b are the same as those in (14). Now we partition u and b after the L th column so that, from (14),

$$(52) \quad x = [u_1 \quad u_2] \begin{bmatrix} b'_1 \\ b'_2 \end{bmatrix} = u_1 b'_1 + u_2 b'_2.$$

We will assume that the columns of u and b have been permuted so that the L factor scores retained for prediction are given by u_1 , or (if one prefers to think of prediction from a rank- L approximation to x) by $u_1 b'_1$. We will now show that the two assumptions are equivalent for prediction problems. Note first, however, that in future samples the weights must be applied to the predictors rather than to the factor scores or to the lower-rank approximation. The latter must be obtained as a row transformation of the prediction matrix, since a prediction equation must be applicable to individual cases.

Let the inverse of b be conformably partitioned and denoted by B' so that

$$(53) \quad B'b = \begin{bmatrix} B'_1 \\ B'_2 \end{bmatrix} [b_1 b_2] = \begin{bmatrix} B'_1 b_1 & B'_1 b_2 \\ B'_2 b_1 & B'_2 b_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Then

$$(54) \quad u_1 = xB_1$$

is a unique solution for u_1 as a transformation on the rows of x . To see this, we let γ be any other such transformation, and let

$$(55) \quad E = \gamma - B_1.$$

Then

$$(56) \quad u_1 = x\gamma = xB_1 + xE = u_1 + xE$$

so that

$$(57) \quad xE = 0,$$

which, since x is basic, implies that E is zero. Now let $\hat{\beta}_u$ be a set of least-squares weights for u_1 . Since u_1 is basic, $\hat{\beta}_u$ is unique. Let $\hat{\beta}_b$ be a set of least-squares weights for $u_1 b'_1$. Since $u_1 b'_1$ is nonbasic, $\hat{\beta}_b$ is not unique. If

$$(58) \quad u_1 b'_1 \hat{\beta}_b - y = \epsilon_b$$

and

$$(59) \quad u_1 \hat{\beta}_u - y = \epsilon_u,$$

the sums of squares of ϵ_b and of ϵ_u will be minimized by $\hat{\beta}_b$ and $\hat{\beta}_u$, respectively. The former sum of squares can be no less than the latter, for we could always take

$$(60) \quad \hat{\beta}_u = b'_1 \hat{\beta}_b.$$

The two sums of squares will be equal if we let

$$(61) \quad \hat{\beta}_b = B_1 \hat{\beta}_u.$$

Therefore, a set of least-squares weights for (58) will be given by $\hat{\beta}_b$ in (61) and

$$(62) \quad \epsilon'_b \epsilon_b = \epsilon'_u \epsilon_u.$$

But since $\hat{\beta}_u$ is unique, $b'_1 \hat{\beta}_b$ must be unique, and (60) holds for all least-squares solutions $\hat{\beta}_b$ of (58). Thus, (58) and (59) are identical, and because of the uniqueness of B_1 in (54), we have

$$(63) \quad \bar{\beta} = B_1 \hat{\beta}_u$$

as a unique set of least-squares weights for x under the assumption of reduced rank.

If it is assumed that the criterion depends solely on the subset of L factors retained for prediction, the general linear hypothesis takes the form

$$(64) \quad y = xB_1\beta_u + e_L,$$

where x , y , and e_L are defined in the first section of this chapter. All of the results of that section may be obtained for the present hypothesis if we replace x by xB , and β by β_u in (1) through (13). In like manner, (48) may be obtained from the derivation of (39). Thus, from (6) and (54) the least-squares estimate of β_u is given by

$$(65) \quad \hat{\beta}_u = (u'_1 u_1)^{-1} u'_1 y = u'_1 y.$$

It has, from (10), the expectation

$$(66) \quad E(\hat{\beta}_u) = \beta_u$$

and, from (11), the covariance matrix

$$(67) \quad C_{\hat{\beta}_u} = \sigma_L^2 (u'_1 u_1)^{-1} = \sigma_L^2 I.$$

An unbiased estimate of the vector of weights to be applied directly to the predictors is given by $\bar{\beta}$ as defined in (63), since

$$(68) \quad E(\bar{\beta}) = E(B_1 \hat{\beta}_u) = B_1 E(\hat{\beta}_u) = B_1 \beta_u.$$

The covariance matrix for these weights will be

$$(69) \quad C_{\bar{\beta}} = E[(B_1 \hat{\beta}_u - B_1 \beta_u)(B_1 \hat{\beta}_u - B_1 \beta_u)'] = B_1 C_{\hat{\beta}_u} B_1' = \sigma_L^2 B_1 B_1'.$$

The estimates of the criterion will now be, from (7),

$$(70) \quad \tilde{y}_L = xB_1 \hat{\beta}_u = x\bar{\beta}.$$

The expected sum of squares for the errors of estimate becomes, from (8),

$$(71) \quad E[(y - \tilde{y}_L)'(y - \tilde{y}_L)] = (N - L)\sigma_L^2.$$

The matrix H for transforming the criterion observations to canonical form may take exactly the same form as in (15):

$$(72) \quad H = (u_1 \ u_2 \ v).$$

The matrix $[u_2 \ v]$ is now arbitrary to the extent that only v was arbitrary before. It will be convenient, however, to define H as in (72). Partitioning the transformed observations somewhat differently from the way it was done in (16), we let

$$(73) \quad z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = Hy = \begin{bmatrix} u_1'y \\ u_2'y \\ v'y \end{bmatrix}.$$

The elements of z_2 and z_3 will all have expected values of zero, while the expectation of z_1 will be

$$(74) \quad E(z_1) = E(u_1'y) = E(\beta_u) = \beta_u.$$

The unbiased estimate for σ_u^2 may be expressed in terms of z_2 and z_3 as

$$(75) \quad \hat{\sigma}_L^2 = \frac{z_2'z_2 + z_3'z_3}{N - L}.$$

The implications of using a reduced-rank solution instead of the conventional solution can perhaps be better understood if the full-rank hypothesis of (1) is retained, rather than the rank- L hypothesis of (64). We first observe that $\bar{\beta}$ is a biased estimate of β , since

$$(76) \quad E(\bar{\beta}) = E(B_1u_1'y) = B_1u_1'\beta = B_1b_1'\beta.$$

Its covariance matrix, which will now be proportional to σ^2 instead of to σ_L^2 , is given by

$$(77) \quad C_{\bar{\beta}} = E[(B_1u_1'y - B_1b_1'\beta)(B_1u_1'y - B_1b_1'\beta)'] = B_1E(u_1'ee'u_1)B_1'$$

since premultiplying (1) by u_1' gives

$$(78) \quad u_1'y = b_1'\beta + u_1'e.$$

Continuing, with (3) in (77),

$$(79) \quad C_{\bar{\beta}} = B_1u_1'E(ee'u_1)B_1' = \sigma^2 B_1B_1'.$$

The first and second moments about β will be

$$(80) \quad E(\bar{\beta} - \beta) = B_1b_1'\beta - \beta = -(I - B_1b_1')\beta = -B_2b_2'\beta$$

and

$$(81) \quad \begin{aligned} E[(\bar{\beta} - \beta)(\bar{\beta} - \beta)'] \\ = C_{\bar{\beta}} + [E(\bar{\beta} - \beta)][E(\bar{\beta} - \beta)]' = \sigma^2 B_1B_1' + B_2b_2'\beta\beta'b_2B_2'. \end{aligned}$$

Equation (11) may be written as

$$(82) \quad C_{\beta} = \sigma^2(x'x)^{-1} = \sigma^2 BB' = \sigma^2 B_1 B_1' + \sigma^2 B_2 B_2'.$$

Thus, from the standpoint of relative efficiency (Mood, 1950, p. 149) in estimating β , $\hat{\beta}$ and $\bar{\beta}$ may be compared in terms of the diagonals of the rightmost terms of (81) and (82). If the trace of the former is smaller, on the average the reduced-rank estimates will be more efficient than the full-rank estimates.

The expected value of z as given by (73) will now be

$$(83) \quad E(z) = \begin{bmatrix} u_1' x \beta \\ u_2' x \beta \\ v' x \beta \end{bmatrix} = \begin{bmatrix} b_1' \beta \\ b_2' \beta \\ 0 \end{bmatrix}.$$

We recall from (19) that $\hat{\beta}$ is computed so that

$$(84) \quad \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} b_1' \hat{\beta} \\ b_2' \hat{\beta} \end{bmatrix}.$$

But $\bar{\beta}$ is computed to reproduce only z_1 :

$$(85) \quad z_1 = u_1' y = b_1' B_1 u_1' y = b_1' \bar{\beta}.$$

We have

$$(86) \quad b_2' \bar{\beta} = b_2' B_1 u_1' y = 0.$$

Thus, the reduced-rank solution, in effect, predicts a value of zero for z_2 rather than a value of $b_2' \hat{\beta}$. If the elements of $b_2' \beta$ are smaller than σ^2 , then the prediction of zero would have the higher relative efficiency.

The statistic $\hat{\sigma}_L^2$ will be an overestimate of σ^2 . To see this, first note that

$$(87) \quad \begin{aligned} E(z_2' z_2 + z_3' z_3) &= \text{tr} [E(z_2 z_2')] + \text{tr} [E(z_3 z_3')] \\ &= \text{tr} (\sigma^2 I + b_2' \beta \beta' b_2) + \text{tr} (\sigma^2 I) \\ &= (M - L) \sigma^2 + \beta' b_2 b_2' \beta + (N - M) \sigma^2 \\ &= (N - L) \sigma^2 + \beta' b_2 b_2' \beta. \end{aligned}$$

Then from (75),

$$(88) \quad E(\hat{\sigma}_L^2) = \sigma^2 + \frac{\beta' b_2 b_2' \beta}{N - L}.$$

Next, we describe the effect of hypothesized rank on our inverse index of weight-efficiency, ψ_{β} . We will denote this index and its estimate by ${}_M \psi_{\bar{\beta}}$ and ${}_M \hat{\psi}_{\bar{\beta}}$, where the full rank M is assumed, and by ${}_L \psi_{\bar{\beta}}$ and ${}_L \hat{\psi}_{\bar{\beta}}$, where the reduced-rank, L , is assumed. Mathematical expectation under the hypothesis of full

rank will be denoted by $E_M(\cdot)$ and under the hypothesis of reduced-rank by $E_L(\cdot)$.

The reduced-rank index ${}_L\psi_{\bar{\beta}}$ was given by (48). To obtain the full-rank index, we first evaluate the rightmost term in (38). Using (81),

$$\begin{aligned}
 (89) \quad E_M[(\beta - \bar{\beta})'x'x(\beta - \bar{\beta})] &= \text{tr} [xE(\bar{\beta} - \beta)(\bar{\beta} - \beta)'x'] \\
 &= \sigma^2 \text{tr} (xB_1B_1'x') + \text{tr} (xB_2b_2'\beta\beta'b_2B_2'x') \\
 &= \sigma^2 \text{tr} (u_1u_1') + \text{tr} (u_2b_2'\beta\beta'b_2u_2') \\
 &= \sigma^2 \text{tr} (u_1'u_1) + \beta'b_2u_2'u_2b_2'\beta \\
 &= L\sigma^2 + \beta'b_2b_2'\beta.
 \end{aligned}$$

Substituting (89) in (38), we obtain

$$(90) \quad {}_M\psi_{\bar{\beta}} = (N + L)\sigma^2 + \beta'b_2b_2'\beta.$$

An unbiased estimate of ${}_L\psi_{\bar{\beta}}$ is, from (75) and (48),

$$(91) \quad {}_L\hat{\psi}_{\bar{\beta}} = (N + L)\hat{\sigma}_L^2 = z_2'z_2 + z_3'z_3 + \frac{2L}{N - L}(z_2'z_2 + z_3'z_3).$$

An unbiased estimate of ${}_M\psi_{\bar{\beta}}$ is, from (87),

$$(92) \quad {}_M\hat{\psi}_{\bar{\beta}} = z_2'z_2 + z_3'z_3 + \frac{2L}{N - M}z_3'z_3.$$

The latter will also be an unbiased estimate of ${}_L\psi_{\bar{\beta}}$, since

$$(93) \quad E_L\left(\frac{z_3'z_3}{N - M}\right) = \sigma_L^2.$$

It would not, however, be as stable an estimate as ${}_L\hat{\psi}_{\bar{\beta}}$, since the rightmost term of (91) is based on more observations than the rightmost term of (92). If ${}_L\hat{\psi}_{\bar{\beta}}$ were used to estimate ${}_M\psi_{\bar{\beta}}$, it would have a positive bias, since, from (88) and (90),

$$(94) \quad E_M({}_L\hat{\psi}_{\bar{\beta}}) = (N + L)\left(\sigma^2 + \frac{\beta'b_2b_2'\beta}{N - L}\right) = {}_M\psi_{\bar{\beta}} + \frac{2L}{N - L}\beta'b_2b_2'\beta.$$

In practice, it would often be convenient to express these estimates in terms of the multiple correlation coefficient. If the metric of the third section is assumed, the elements of z_1 and z_2 will be the correlations between the factor scores and the criterion, or factor validities. Since the factor scores are uncorrelated, the squared multiple correlation between the first L factors and the criterion will be

$$(95) \quad R_L^2 = z_1'z_1 = 1 - z_2'z_2 - z_3'z_3.$$

Hence (91) and (92) are equivalent to

$$(96) \quad {}_L\hat{\psi}_{\beta} = 1 - R_L^2 + \frac{2L(1 - R_L^2)}{N - L},$$

and

$$(97) \quad {}_M\hat{\psi}_{\beta} = 1 - R_L^2 + \frac{2L(1 - R_M^2)}{N - M}.$$

Equation (96) is, of course, equivalent to (49). Although ${}_L\hat{\psi}_{\beta}$ and ${}_M\hat{\psi}_{\beta}$ will in general differ only very slightly, the former is to be preferred in applications, since R_L will be less inflated by overfit than will R_M .

In theoretical comparisons of different factor solutions, ${}_M\hat{\psi}_{\beta}$ will be most useful, since it is a function of the loadings of the discarded factors. The optimal factor solution would be that which minimized the rightmost term of equation (90).

Some Particular Reduced Rank Procedures

Of the five particular rank-reduction procedures considered in the present study, three involve prediction from principal-axes factors, and two involve prediction from a subset of the original predictors. Summerfield and Lubin (1951) have shown that a subset of predictors is equivalent to a subset of orthogonal triangular (or square-root) factor scores. The first factor is simply the first predictor. The second factor is that portion of the second predictor which cannot be predicted from the first. The third factor is that portion of the third predictor which cannot be predicted from the first and second. The remaining factors are similarly obtained. Each factor will thus be independent of the earlier factors and of the predictors corresponding to them, and will therefore have zero loadings on those predictors. Accordingly, the factor-loading matrix will be a lower triangular matrix, i.e., its supra-diagonal elements will all be zero.

The predictor-selection and predictor-elimination methods may be thought of as procedures for placing the predictors in the approximate order of their contribution to the multiple correlation with the criterion. Since the triangular factors are determined by the ordering of the predictors, the first L factors will tend to give the highest multiple correlation obtainable with a subset of L predictors.

Prediction from the principal-axes factors giving the highest validity is similar to these methods in that the subset of factors to be retained is entirely determined by the characteristics of the sample from which regression weights are to be computed. Under these circumstances, none of the indices of validity or weight-validity is directly applicable, since all are based on the assumption that, for given L , the subset of predictors to be retained is determined in advance of observing the criterion. A detailed analysis of the con-

sequences of choosing factors on the basis of the observed y will not be attempted. Clearly, however, the fewer the degrees of freedom available, the larger will be the variance of the sample validities, and the smaller the probability that the subset of L factors having the largest true validity will give the largest sample validity. Moreover, the true validity for the subset chosen would tend to fall short of the true validity for the optimal subset, and the sample validity for the chosen subset would tend to overestimate its true validity, in inverse proportion to the degrees of freedom. Still, it seems that subsets of predictors selected in this way would usually have higher true validities than would arbitrarily chosen predictors.

Although the foregoing discussion is not concrete enough to lead to precise conclusions, it does suggest the desirability of having a method of factoring that would provide an a priori expectation as to the contributions to validity of the individual factors. The success of using approximation to the intercorrelation matrix or to its inverse as a criterion for selecting predictors will in part be determined by the extent to which contribution to the approximation is related to contribution to validity.

In describing the two particular factor methods in terms of the general model of the preceding section, we will consider first the triangular factors. For the general factor-loading matrix, b , we substitute a lower triangular factor-loading matrix, t . But where b was partitioned only after the L th column, we will partition t also after the L th row, so that

$$(98) \quad t = \begin{bmatrix} t_1 & t_2 \end{bmatrix} = \begin{bmatrix} t_{11} & 0 \\ t_{12} & t_{22} \end{bmatrix}.$$

We will partition the inverse of t similarly, and denote it by T' . It may be readily verified that

$$(99) \quad T' = \begin{bmatrix} T'_1 \\ T'_2 \end{bmatrix} = \begin{bmatrix} t_{11}^{-1} & 0 \\ -t_{22}^{-1}t_{21}t_{11}^{-1} & t_{22}^{-1} \end{bmatrix} = t^{-1}.$$

It will also be convenient to partition the predictor matrix x after the L th column, and to partition the regression vectors β and $\bar{\beta}$ after the L th element.

We first note, from (52), that

$$(100) \quad x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} = u_1 t'_1 + u_2 t'_2 = \begin{bmatrix} u_1 t'_{11} & u_1 t'_{12} \end{bmatrix} + \begin{bmatrix} 0 & u_2 t'_{22} \end{bmatrix}.$$

Thus

$$(101) \quad u_1 t'_1 = \begin{bmatrix} x_1 & u_1 t'_{12} \end{bmatrix}$$

and

$$(102) \quad x_2 = u_1 t'_{12} + u_2 t'_{22}.$$

The first term on the right of (102) is that portion of x_2 which can be predicted

from x_1 , while the second term is that portion of x_2 which is independent of x_1 . Thus the "reduced-rank approximation" of x on which predictions are based is from (101) composed simply of the retained predictors augmented by the portion of the discarded predictors that is determined by those retained.

From (63) and (65), the estimated regression weights will be

$$(103) \quad \bar{\beta} = T_1 u'_1 y = \begin{bmatrix} (t'_{11})^{-1} u'_1 y \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{\beta}_1 \\ \bar{\beta}_2 \end{bmatrix}.$$

Their expected values, under the full-rank hypothesis, will be, from (76)

$$(104) \quad E(\bar{\beta}) = T_1 t'_1 \beta = \begin{bmatrix} (t'_{11})^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} t'_{11} & t'_{21} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \beta_1 + (t'_{11})^{-1} t'_{21} \beta_2 \\ 0 \end{bmatrix} = \begin{bmatrix} E(\bar{\beta}_1) \\ E(\bar{\beta}_2) \end{bmatrix}.$$

The value for $E(\bar{\beta}_1)$ in (104) may be thought of as an expression for the optimal weights for a subset of predictors in terms of the optimal weights for the entire set. The original weights for the retained predictors are altered as a function of the original weights for the discarded predictors. This illustrates the point made in the section on accuracy of predictions, to the effect that weights for a subset of predictors cannot be properly evaluated in terms of how closely they approximate the weights for the entire set. The covariance matrix of the sample regression weights, obtained from (79), is

$$(105) \quad C_{\bar{\beta}} = \sigma^2 T_1 T'_1 = \sigma^2 \begin{bmatrix} (t'_{11})^{-1} t'_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

The expected values of the transformed criterion observations will be, from (83),

$$(106) \quad E(z) = \begin{bmatrix} E(z_1) \\ E(z_2) \\ E(z_3) \end{bmatrix} = \begin{bmatrix} t'_1 \beta \\ t'_2 \beta \\ 0 \end{bmatrix} = \begin{bmatrix} t'_{11} \beta_1 + t'_{21} \beta_2 \\ t'_{22} \beta_2 \\ 0 \end{bmatrix}.$$

From (90), the inverse index of weight efficiency ${}_M \psi_{\bar{\beta}}$ is given by

$$(107) \quad {}_M \psi_{\bar{\beta}} = (N + L) \sigma^2 + \beta' t'_2 t'_2 \beta = (N + L) \sigma^2 + \beta'_2 t'_{22} t'_{22} \beta_2.$$

To obtain the principal-axes solution, we first express the predictor matrix x in terms of its basic structure (Horst, 1961, ch. 17):

$$(108) \quad x = P \Delta Q'.$$

Now, in place of the general factor-score matrix u we have the principal-axes factor-score matrix P . The principal-axes factor-loading matrix, corresponding to the general b is given by $Q \Delta$, where Q is a square orthonormal and Δ a diagonal matrix. Equation (50) now takes the form

$$(109) \quad x'x = Q \Delta^2 Q'.$$

The eigenvalues and eigenvectors of $x'x$ will be given by the elements of Δ^2 and the columns of Q respectively. We may partition the factors on the right of (108) to obtain

$$\begin{aligned}
 x &= [P_1 \ P_2] \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \begin{bmatrix} Q'_1 \\ Q'_2 \end{bmatrix} \\
 (110) \quad &= [P_1 \ P_2] \begin{bmatrix} \Delta_1 Q'_1 \\ \Delta_2 Q'_2 \end{bmatrix} \\
 &= P_1 \Delta_1 Q'_1 + P_2 \Delta_2 Q'_2.
 \end{aligned}$$

As before, both the factor-score and factor-loading matrices are considered to be partitioned after the L th column. For the inverse of the factor-loading matrix, B' , we will now have

$$(111) \quad [Q_1 \Delta_1 \quad Q_2 \Delta_2]^{-1} = \begin{bmatrix} \Delta_1^{-1} Q'_1 \\ \Delta_2^{-1} Q'_2 \end{bmatrix}.$$

The sample regression vector is, from (63) and (65),

$$(112) \quad \tilde{\beta} = Q_1 \Delta_1^{-1} P'_1 y.$$

Under the full-rank hypothesis, the lower-rank sample regression weights will have the covariance matrix, from (79),

$$(113) \quad C_{\tilde{\beta}} = \sigma^2 Q_1 \Delta_1^{-2} Q'_1.$$

From (83), the canonical form of the criterion will have the expectation

$$(114) \quad E(z) = \begin{bmatrix} E(z_1) \\ E(z_2) \\ E(z_3) \end{bmatrix} = \begin{bmatrix} \Delta_1 Q'_1 \beta \\ \Delta_2 Q'_2 \beta \\ 0 \end{bmatrix}.$$

Equation (90) will now take the form

$$(115) \quad {}_M \psi_{\tilde{\beta}} = (N + L) \sigma^2 + \beta' Q_2 \Delta_2^2 Q'_2 \beta.$$

The specific reduced-rank prediction models may be obtained from the foregoing development by assuming appropriate permutations either of the predictors, in the case of triangular factors, or of the columns of P and Q , and of the elements of Δ , in the case of principal-axes factors. We note from (73) and (83) that each element of z_1 and z_2 is determined by only one factor: the observed value by the factor scores, the expected value by the factor loadings. In predictor selection, each time a predictor is selected, a factor, and hence an element of z_1 , is determined. At each step in the procedure,

that predictor is selected which will make the next element of z_1 as large (in absolute value) as possible. In predictor elimination, a factor and hence an element of z_2 , is determined each time a predictor is eliminated. At each step, that predictor is eliminated which will make the next element of z_2 as small (in absolute value) as possible.

In the method of predicting from the factors giving the best least-squares approximation to the predictor intercorrelation matrix, the elements of Δ are placed in order from largest to smallest, so that the largest are in Δ_1 and the smallest in Δ_2 . If the inverse is to be approximated, the elements of Δ are placed in the opposite order, i.e., from smallest to largest. (When we speak of ordering the elements of Δ , we assume, of course, that the columns of P and Q are permuted correspondingly.) In the method of predicting from the principal-axes factors giving the highest validity, the factors are permuted so as to place the elements of z_1 and z_2 in order of absolute value from largest to smallest, with the largest values in z_1 , the smallest in z_2 .

The Problem of Finding an Optimal Reduced-Rank Solution

There are three major problems involved in obtaining an optimal reduced-rank solution. The first concerns the method of rank reduction: whether subsets of the original predictors, of the principal-axes factors, or of factors obtained by some other method will give the most accurate prediction in future samples. The second problem is, having obtained the factors, to specify the subset of a given size that may be expected to provide the greatest accuracy of prediction. The third problem is, having specified the subset which would be used for any given rank, to determine the particular rank that will tend to lead to the most accurate predictions.

The estimate of the inverse index of weight-efficiency given in (91) and (96) provides a solution (or a potential solution) to the third problem. It does not, however, enhance our ability to deal with the second problem, since, as can be seen from (96), it merely indicates the traditional approach; namely, to attempt to select that subset of predictors of given size having the highest multiple correlation with the criterion. The drawbacks of such an approach when degrees of freedom are limited were discussed in the preceding section. Since a reduced-rank solution is indicated only when degrees of freedom are limited, a selection method that is independent of the criterion might well be preferable. Some evidence favoring this view is provided in the empirical portion of the present study. In the present section we assume that view to be correct and accordingly consider only methods of selection which are independent of the criterion.

If the present analysis is correct, an optimal solution will be one which minimizes ${}_M\psi_{\beta}$ as given in (90). In the absence of observations on the criterion, nothing can be said about β or σ^2 , so our only course is to seek a value for b_2 which will minimize $\beta' b_2 b_2' \beta$ for general β . The quantity to be minimized

may also be expressed as the sum of squares of the expected values of the z_2 , as given in (83):

$$(116) \quad \beta' b_2 b_2' \beta = [E(z_2)]' [E(z_2)].$$

Minimizing this quantity will be equivalent to making the elements of $E(z_2)$ as small (in absolute value) as possible. We let the i th element of

$$(117) \quad \bar{z} = \begin{bmatrix} E(z_1) \\ E(z_2) \end{bmatrix}$$

be denoted by \bar{z}_i . If we knew these values, the second of the problems stated above would be solved by discarding those factors for which \bar{z}_i was smallest. Denoting the column of factor loadings for the i th factor by $b_{.i}$, we have, from (83),

$$(118) \quad \bar{z}_i = b_{.i}' \beta.$$

Let D be a diagonal matrix whose i th element is given by

$$(119) \quad D_i = \sqrt{b_{.i}' b_{.i}}.$$

Let

$$(120) \quad W = b D^{-1}.$$

Denoting the i th column of W by $W_{.i}$, we have

$$(121) \quad W_{.i}' W_{.i} = \frac{b_{.i}' b_{.i}}{b_{.i}' b_{.i}} = 1.$$

The expected values of z_1 and z_2 can now be expressed in terms of D and W as

$$(122) \quad \bar{z} = b' \beta = D W' \beta,$$

or

$$(123) \quad \bar{z}_i = D_i W_{.i}' \beta.$$

Since we have assumed that nothing is known about β , and since (121) holds for all i , we can have no a priori expectation as to the magnitude of $W_{.i}' \beta$. Thus our only basis for predicting the rank order of the \bar{z}_i in the absence of criterion observations will be the magnitudes of the D_i . A tentative solution for the problem of which factors to retain for prediction, then, will be to discard those factors having the smallest values of D_i . From (119), we see that D_i^2 is the sum of squares of the loadings for the i th factor, or the variance accounted for by that factor. Thus, for a rank- L solution, we wish to retain those L factors giving the best least-squares approximation to the predictor matrix.

It is well known that the principal-axes factors will give a better least-squares approximation to the predictor matrix than will factors obtained

by any other method. Thus, as a tentative answer to the first of the above problems we obtain the principal-axes solution.

Now, given the restriction that the factors be selected independently of the criterion, we can state that the best prediction possible with a reduced-rank solution will be obtained from the principal-axes factors giving the best least-squares approximation to the correlation matrix. We note that, for a principal-axes solution, D and W become the Δ and Q of the preceding section. Thus we can also state that the method of approximating the inverse will give the worst possible predictions, since with that method one discards the factors corresponding to the largest elements of Δ .

We have shown that, with appropriate assumptions, the principal-axes factors making the largest contribution to the variance of the predictors (or simply, the largest principal-axes factors) are optimal with respect to our index of expected accuracy of prediction. It may be shown that the factors are also optimal with respect to the variance of the sample regression weights. The sum of these variances will be smaller than for any other method of rank reduction. From (69) (or (79)), this sum will be proportional to the trace of $B_1 B_1'$. We let

$$(124) \quad g' = B u' = B_1 u_1' + B_2 u_2',$$

so that

$$(125) \quad g' - B_2 u_2' = B_1 u_1'.$$

It is well known that

$$(126) \quad \text{tr}(u_1 B_1' B_1 u_1') = \text{tr}(B_1 B_1')$$

will be a minimum when B_2 is composed of the largest principal-axes factors of

$$(127) \quad g' g = B B' = (x' x)^{-1} = Q \Delta^{-2} Q'.$$

Equivalently, the above trace will be a maximum when b_1 is composed of the largest principal-axes factors of $x' x$.

The major conclusion of this section is that, in the absence of criterion observations, the best index to use for selection of predictors or factors will be the amount of variance accounted for in the predictor data matrix. In the case where a subset of the original predictors is to be used, one would eliminate those predictors for which the trace of $t_{22} t_{22}'$ in (107) is a minimum. Where a factor solution is feasible, the largest principal-axes factors would be retained. The important question of how many degrees of freedom must be available before the criterion observations can be used to advantage in the selection process has been left open. Thus a sound basis for deciding whether to use the methods above or to use methods which attempt to maximize the sample multiple correlation with the criterion is still lacking.

CHAPTER 3

AN EMPIRICAL COMPARISON OF FIVE REDUCED RANK PROCEDURES

The Data

A typical application of regression methods is to the problem of predicting academic success as measured by college grades. The data for the present comparisons were taken from a recent study of academic prediction by Shanker (1961). Twenty-nine predictor variables and five separate criterion variables are used. Fifteen of the predictors are those composing the University of Washington Entrance Battery. These have been in use for predicting college grades since 1953, and include age, sex, test scores, and high-school grades. The remaining predictors are taken from the Edwards Personal Preference Schedule (EPPS). The 15 variables of the EPPS are ipsative; i.e., any one can be computed exactly from the remaining 14. Accordingly, only 14 are used here, since the 15th would be completely redundant for purposes of prediction. The EPPS variables are described by Edwards (1954). Descriptions of the Entrance Battery variables are given by Shanker (1961). Since the specific nature of the predictors is not of immediate interest in the present study, we simply list them here.

Edwards Personal Preference Schedule Variables

- | | |
|-----------------|---------------------|
| 1. Achievement | 8. Succorance |
| 2. Deference | 9. Dominance |
| 3. Order | 10. Abasement |
| 4. Exhibition | 11. Nurturance |
| 5. Autonomy | 12. Change |
| 6. Affiliation | 13. Endurance |
| 7. Intraception | 14. Heterosexuality |

High-School Grade-Point Averages

- | | |
|----------------------|---------------------|
| 15. English | 18. Social Science |
| 16. Mathematics | 19. Natural Science |
| 17. Foreign Language | 20. Electives |

Test Scores

- | | |
|--------------------------|----------------------------|
| 21. Vocabulary | 25. Mathematics |
| 22. Mechanical Knowledge | 26. Social Science |
| 23. English Usage | 27. Quantitative Reasoning |
| 24. English Spelling | |

Other Variables

- 28. Age
- 29. Sex (coded 0 for male, 1 for female)

The criterion variables consist of grade-point averages in various college course areas. The five criteria chosen for the present study were those having 500 or more cases available, as listed below.

- 1. All-University, 973 cases
- 2. Mathematics, 541 cases
- 3. English Composition, 804 cases
- 4. Chemistry, 526 cases
- 5. Psychology, 507 cases

The cases used were 973 students who entered the University of Washington as freshmen between 1953 and 1958. Only those students were included for whom measurements on all predictors and at least one criterion variable were available. Scores on the criterion variables and on the Entrance Battery (predictors 15-29) were obtained from the files of the University of Washington Division of Counseling and Testing Services. The EPPS data (predictors 1-14) were obtained partly from Edwards, partly from Wright (1957), and largely from the Division of Counseling and Testing Services files.

Method

The five reduced-rank prediction methods chosen for comparison were the following.

- 1. The predictor-elimination method (Horst and MacEwan, 1960)
- 2. Predictor selection by the accretion method (Horst, 1955)
- 3. The method of largest principal-axes factors (Horst, 1941)
- 4. The method of smallest principal-axes factors (Guttman, 1958)
- 5. The method using the principal-axes factors giving the highest multiple correlation.

As noted in the introduction, we can be virtually certain that, for sufficiently small samples, one or more of these methods will give more accurate predictions than will the standard full-rank method. And as shown in the last section of Chapter 2, there is reason to believe that method 3 will be superior to the others for samples below some critical size. Similarly, method 4 would be expected to give the poorest predictions. We would expect also that the statistics ${}_L\hat{\psi}_{\beta}$ as given by (91) and \hat{W} as given by (46) would give some indication of the accuracy of prediction in future samples obtainable from a particular set of weights.

The method used for the empirical comparisons consisted essentially of replications of the following procedure. All cases with measurements available on a particular criterion were taken as the statistical population. From this population a random sample was drawn. Regression weights were computed

for each reduced-rank method for each rank from 1 to 29. Thus 29 sets of weights for each method were computed. The sets of weights for rank 29 were, of course, the same (aside from rounding error) for all methods. From the cases remaining in the population after the original sample was removed, a new random sample was drawn. Each set of weights computed in the original sample was then applied to the new sample, and measures of accuracy of prediction were computed. For all computations, predictor and criterion variables were normalized as described in the second section of Chapter 2. In effect, then, means and sums of squares were equated for all variables on all samples. Differences in these values, therefore, do not show up in the total squared errors of prediction.

For each of the five criterion variables, this design, using all five reduced-rank methods, was replicated for six different original-sample sizes: 255, 210, 165, 120, 75, and 30 cases. The new samples consisted of 252 cases for all replications. Weight-validities were used as measures of accuracy of prediction.

An additional set of replications was carried out for criterion 1 (All-University) only, and omitting method 4. Here the estimates of weight-validity and of total squared errors of prediction were also computed from the original samples. A wider range of original-sample sizes was used: the six sizes above and also sizes of 435, 390, 345, and 300 cases. A second new sample was randomly drawn for each replication from the cases remaining in the population after the original sample and the first new sample were removed. Both new samples again consisted of 252 cases for all replications. As measures of accuracy of prediction when the original sample weights were applied to each of the two new samples, total squared errors of prediction as well as weight-validities were computed.

All phases of the above procedures were carried out on the IBM 709 computer, using programs written especially for this study. The method of drawing the samples was as follows. The cases in a particular criterion population of, say, NT students were assigned sequential numbers from 1 to NT . A sequence of random numbers was generated using a procedure described in the *WDPC Users Manual* (Western Data Processing Center, 1961, sec. 9.2.4). The original sample of size N_0 consisted of the cases corresponding to the first N_0 distinct numbers modulo NT from the sequence of random numbers. The remaining $NT - N_0$ cases were renumbered sequentially from 1 to $NT - N_0$. The new sample of size N_1 consisted of the first N_1 distinct numbers modulo $NT - N_0$ from a second sequence of random numbers. In a similar way, all other samples were obtained, using a new sequence of random numbers for each sample.

After obtaining the original sample, the matrix of predictor intercorrelations and the vector of the correlations between the predictors and the criterion were computed. Retaining the notation of the preceding chapter and recalling that the variables in x and y were normalized, the predictor

intercorrelation matrix was computed by (25) and the vector of predictor-criterion correlations by

$$(128) \quad r_c = x'y.$$

Next the predictor elimination and predictor selection procedures were carried out and the corresponding regression weights computed, using the procedures described by Horst and MacEwan (1960) and by Horst (1955), respectively. The matrix r was then factored as in (109). The regression weights for the three principal-axes methods were computed as follows. We let z_L denote the L th element of z_i , $Q_{.L}$ denote the L th column of Q_1 and Δ_L the L th element of Δ_1 .

First the vector of factor validities z_1 was computed from

$$(129) \quad z_1 = \Delta_1^{-1}Q_1'r_c.$$

Equation (129) is equivalent to (73), since, from (108), (110), and (128),

$$(130) \quad \Delta_1^{-1}Q_1'r_c = \Delta_1^{-1}Q_1'x'y = \Delta_1^{-1}Q_1'(Q_1\Delta_1P_1' + Q_2\Delta_2P_2')y = P_1'y.$$

The regression vector for rank L was computed by

$$(131) \quad \bar{\beta}_L = Q_1\Delta_1^{-1}z_1 = \sum_{i=1}^L Q_{.i}\Delta_i^{-1}z_i,$$

which, it may be noted, is equivalent to (112). Thus the regression vector for rank $L + 1$ was obtained from the vector for rank L by

$$(132) \quad \bar{\beta}_{L+1} = \bar{\beta}_L + Q_{.L+1}\Delta_{L+1}^{-1}z_{L+1}.$$

The weights for methods 3, 4, and 5 were all computed in the same way, the only difference being in the order of summation.

The new sample was drawn and the various correlations computed as for the original sample. The weight-validity and total squared errors of prediction obtained with a particular vector of weights were computed respectively by

$$(133) \quad W = \frac{r'_c\bar{\beta}_L}{\sqrt{\bar{\beta}'_L r \bar{\beta}_L}}$$

and

$$(134) \quad \psi = 1 - 2r'_c\bar{\beta}_L + \bar{\beta}'_L r \bar{\beta}_L.$$

Equations (133) and (134) are, of course, equivalent to (42) and (43). Note that r and r_c in (133) and (134) are computed on the new sample while $\bar{\beta}_L$ was computed on the original sample.

Results and Discussion

The weight-validities obtained with methods 1, 2, 3, and 5 on all five criteria are given in Table 1. The six pages of Table 1 correspond to the

TABLE 1
Weight-Validities for Four Methods and Five Criteria
($N_0 = 255$)

| Criteria: Methods: | All-Univ | | | | | Math | | | | | Engl Comp | | | | | Chem | | | | | Psych | | | | |
|-----------------------|----------|-----|-----|-----|--|------|-----|-----|-----|--|-----------|-----|-----|-----|--|------|-----|-----|-----|--|-------|-----|-----|-----|--|
| | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | |
| 1 | 455 | 455 | 551 | 551 | | 305 | 390 | 414 | 414 | | 462 | 547 | 640 | 640 | | 401 | 404 | 441 | 441 | | 406 | 406 | 416 | 416 | |
| 2 | 484 | 484 | 568 | 576 | | 375 | 382 | 416 | 407 | | 615 | 676 | 616 | 613 | | 372 | 459 | 446 | 496 | | 477 | 475 | 468 | 468 | |
| 3 | 536 | 536 | 569 | 591 | | 415 | 415 | 416 | 400 | | 607 | 645 | 580 | 618 | | 448 | 426 | 411 | 473 | | 489 | 481 | 469 | 487 | |
| 4 | 529 | 529 | 571 | 595 | | 421 | 421 | 416 | 401 | | 616 | 646 | 608 | 665 | | 448 | 451 | 411 | 460 | | 489 | 504 | 482 | 488 | |
| 5 | 521 | 521 | 577 | 555 | | 435 | 411 | 421 | 418 | | 645 | 653 | 659 | 640 | | 418 | 422 | 393 | 450 | | 492 | 511 | 487 | 488 | |
| 6 | 522 | 522 | 575 | 530 | | 422 | 412 | 421 | 414 | | 661 | 637 | 666 | 643 | | 409 | 412 | 399 | 451 | | 509 | 507 | 485 | 488 | |
| 7 | 498 | 498 | 575 | 529 | | 404 | 396 | 414 | 426 | | 663 | 634 | 669 | 651 | | 389 | 389 | 463 | 437 | | 510 | 497 | 485 | 486 | |
| 8 | 494 | 494 | 577 | 531 | | 392 | 383 | 417 | 426 | | 661 | 627 | 644 | 623 | | 389 | 393 | 391 | 426 | | 501 | 489 | 482 | 484 | |
| 9 | 494 | 494 | 572 | 529 | | 393 | 393 | 405 | 419 | | 661 | 622 | 644 | 608 | | 385 | 413 | 390 | 431 | | 492 | 486 | 481 | 480 | |
| 10 | 488 | 491 | 567 | 530 | | 374 | 374 | 416 | 417 | | 648 | 624 | 648 | 609 | | 392 | 406 | 380 | 417 | | 477 | 469 | 486 | 463 | |
| 11 | 496 | 488 | 566 | 524 | | 371 | 371 | 416 | 407 | | 635 | 629 | 630 | 608 | | 399 | 412 | 377 | 406 | | 481 | 475 | 500 | 473 | |
| 12 | 490 | 496 | 564 | 532 | | 368 | 368 | 412 | 399 | | 634 | 626 | 635 | 608 | | 397 | 410 | 375 | 410 | | 475 | 468 | 500 | 468 | |
| 13 | 490 | 492 | 553 | 527 | | 375 | 375 | 414 | 395 | | 636 | 627 | 635 | 633 | | 411 | 419 | 377 | 411 | | 478 | 481 | 498 | 470 | |
| 14 | 486 | 487 | 553 | 524 | | 372 | 372 | 406 | 395 | | 637 | 626 | 638 | 635 | | 406 | 418 | 376 | 411 | | 485 | 485 | 499 | 473 | |
| 15 | 489 | 498 | 544 | 508 | | 371 | 371 | 400 | 389 | | 635 | 637 | 640 | 637 | | 405 | 412 | 385 | 415 | | 486 | 481 | 499 | 471 | |
| 16 | 485 | 498 | 541 | 511 | | 369 | 369 | 406 | 380 | | 625 | 637 | 642 | 640 | | 414 | 409 | 367 | 412 | | 477 | 481 | 505 | 468 | |
| 17 | 482 | 500 | 575 | 515 | | 372 | 372 | 404 | 385 | | 628 | 638 | 643 | 641 | | 413 | 408 | 372 | 410 | | 474 | 483 | 508 | 470 | |
| 18 | 483 | 499 | 577 | 514 | | 376 | 376 | 404 | 385 | | 629 | 635 | 644 | 638 | | 408 | 415 | 365 | 406 | | 470 | 478 | 490 | 468 | |
| 19 | 483 | 502 | 551 | 511 | | 379 | 379 | 403 | 385 | | 628 | 631 | 641 | 639 | | 412 | 415 | 363 | 408 | | 471 | 474 | 481 | 466 | |
| 20 | 479 | 499 | 551 | 505 | | 384 | 384 | 402 | 386 | | 631 | 632 | 642 | 639 | | 410 | 410 | 417 | 410 | | 470 | 470 | 483 | 473 | |
| 21 | 490 | 496 | 545 | 501 | | 383 | 383 | 408 | 381 | | 636 | 632 | 638 | 640 | | 407 | 408 | 413 | 415 | | 476 | 476 | 482 | 474 | |
| 22 | 493 | 497 | 541 | 499 | | 381 | 381 | 405 | 381 | | 638 | 632 | 638 | 642 | | 403 | 404 | 413 | 414 | | 478 | 478 | 481 | 472 | |
| 23 | 494 | 494 | 522 | 502 | | 382 | 382 | 395 | 384 | | 636 | 633 | 639 | 639 | | 407 | 408 | 413 | 413 | | 478 | 478 | 482 | 470 | |
| 24 | 497 | 495 | 524 | 500 | | 383 | 383 | 399 | 385 | | 635 | 634 | 635 | 639 | | 411 | 408 | 412 | 407 | | 477 | 477 | 483 | 471 | |
| 25 | 498 | 498 | 521 | 500 | | 384 | 384 | 393 | 381 | | 636 | 634 | 636 | 636 | | 408 | 412 | 414 | 409 | | 477 | 477 | 485 | 474 | |
| 26 | 498 | 498 | 506 | 502 | | 384 | 384 | 393 | 382 | | 636 | 636 | 638 | 637 | | 409 | 409 | 410 | 412 | | 476 | 476 | 485 | 473 | |
| 27 | 499 | 499 | 507 | 501 | | 385 | 385 | 392 | 383 | | 637 | 639 | 636 | 637 | | 410 | 411 | 409 | 411 | | 477 | 477 | 482 | 473 | |
| 28 | 501 | 501 | 507 | 502 | | 384 | 384 | 384 | 384 | | 637 | 637 | 636 | 637 | | 410 | 410 | 411 | 410 | | 477 | 477 | 481 | 474 | |
| 29 | 500 | | | | | 383 | | | | | 637 | | | | | 410 | | | | | 477 | | | | |
| R_0 | 659 | | | | | 539 | | | | | 705 | | | | | 623 | | | | | 626 | | | | |
| R_1 | 667 | | | | | 515 | | | | | 770 | | | | | 557 | | | | | 580 | | | | |

Decimal point preceding each entry has been omitted.

TABLE 1 (Cont.)
Weight-Validities for Four Methods and Five Criteria
($N_0 = 210$)

| Criteria: Methods | All-Univ | | | | | Math | | | | | Engl Comp | | | | | Chem | | | | | Psych | | | | |
|----------------------|----------|-----|-----|-----|--|------|-----|-----|-----|--|-----------|-----|-----|-----|--|------|-----|-----|-----|--|-------|-----|-----|-----|--|
| | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | |
| 1 | 407 | 407 | 478 | 478 | | 361 | 300 | 464 | 464 | | 499 | 499 | 543 | 543 | | 432 | 432 | 463 | 463 | | 353 | 353 | 409 | 409 | |
| 2 | 462 | 460 | 491 | 479 | | 439 | 382 | 465 | 477 | | 540 | 540 | 543 | 546 | | 454 | 471 | 466 | 450 | | 445 | 445 | 480 | 480 | |
| 3 | 459 | 490 | 491 | 484 | | 440 | 440 | 469 | 451 | | 596 | 596 | 547 | 596 | | 488 | 481 | 473 | 482 | | 443 | 443 | 483 | 477 | |
| 4 | 439 | 474 | 501 | 491 | | 433 | 433 | 448 | 440 | | 627 | 627 | 596 | 594 | | 528 | 508 | 473 | 481 | | 461 | 461 | 488 | 463 | |
| 5 | 479 | 473 | 504 | 490 | | 424 | 424 | 429 | 431 | | 614 | 614 | 601 | 568 | | 523 | 530 | 478 | 457 | | 475 | 475 | 492 | 469 | |
| 6 | 451 | 466 | 497 | 504 | | 411 | 411 | 418 | 440 | | 620 | 620 | 602 | 573 | | 527 | 525 | 483 | 477 | | 466 | 466 | 489 | 448 | |
| 7 | 445 | 463 | 500 | 492 | | 374 | 374 | 415 | 435 | | 628 | 628 | 602 | 589 | | 518 | 533 | 485 | 486 | | 461 | 461 | 487 | 425 | |
| 8 | 456 | 480 | 488 | 505 | | 375 | 375 | 394 | 446 | | 623 | 635 | 607 | 607 | | 513 | 525 | 484 | 494 | | 452 | 452 | 491 | 436 | |
| 9 | 456 | 491 | 487 | 513 | | 362 | 362 | 383 | 416 | | 622 | 630 | 619 | 618 | | 516 | 522 | 485 | 503 | | 460 | 438 | 491 | 432 | |
| 10 | 461 | 470 | 488 | 503 | | 365 | 361 | 380 | 395 | | 616 | 628 | 620 | 607 | | 526 | 527 | 485 | 506 | | 460 | 428 | 499 | 437 | |
| 11 | 448 | 484 | 491 | 504 | | 377 | 363 | 382 | 395 | | 618 | 625 | 607 | 614 | | 519 | 524 | 487 | 500 | | 446 | 436 | 503 | 434 | |
| 12 | 465 | 483 | 491 | 498 | | 384 | 375 | 393 | 383 | | 623 | 618 | 623 | 613 | | 527 | 526 | 486 | 502 | | 439 | 439 | 511 | 428 | |
| 13 | 472 | 472 | 491 | 497 | | 382 | 382 | 412 | 382 | | 628 | 621 | 633 | 616 | | 533 | 526 | 486 | 505 | | 437 | 437 | 488 | 439 | |
| 14 | 473 | 473 | 485 | 494 | | 374 | 379 | 401 | 372 | | 625 | 625 | 630 | 616 | | 533 | 531 | 487 | 511 | | 439 | 433 | 490 | 437 | |
| 15 | 478 | 478 | 481 | 489 | | 366 | 370 | 396 | 363 | | 618 | 618 | 629 | 613 | | 533 | 530 | 466 | 515 | | 445 | 438 | 490 | 442 | |
| 16 | 486 | 486 | 481 | 484 | | 357 | 361 | 392 | 353 | | 622 | 622 | 628 | 609 | | 532 | 530 | 486 | 518 | | 448 | 441 | 492 | 442 | |
| 17 | 485 | 485 | 486 | 485 | | 358 | 355 | 395 | 348 | | 620 | 620 | 627 | 613 | | 528 | 535 | 486 | 517 | | 445 | 445 | 488 | 437 | |
| 18 | 481 | 481 | 486 | 481 | | 358 | 352 | 395 | 340 | | 628 | 628 | 627 | 616 | | 526 | 530 | 520 | 516 | | 446 | 446 | 488 | 434 | |
| 19 | 479 | 484 | 499 | 481 | | 355 | 354 | 395 | 343 | | 628 | 628 | 621 | 617 | | 525 | 532 | 521 | 518 | | 449 | 449 | 486 | 440 | |
| 20 | 478 | 482 | 502 | 482 | | 354 | 351 | 393 | 347 | | 628 | 628 | 622 | 613 | | 525 | 528 | 513 | 521 | | 447 | 450 | 488 | 444 | |
| 21 | 476 | 484 | 497 | 484 | | 358 | 350 | 377 | 346 | | 625 | 625 | 622 | 612 | | 522 | 531 | 513 | 518 | | 441 | 448 | 485 | 446 | |
| 22 | 477 | 482 | 494 | 483 | | 355 | 348 | 357 | 342 | | 618 | 618 | 602 | 621 | | 519 | 530 | 511 | 517 | | 445 | 445 | 478 | 445 | |
| 23 | 479 | 479 | 480 | 481 | | 349 | 348 | 357 | 340 | | 620 | 620 | 617 | 621 | | 516 | 527 | 520 | 517 | | 445 | 445 | 481 | 444 | |
| 24 | 479 | 479 | 481 | 484 | | 346 | 352 | 359 | 341 | | 620 | 620 | 614 | 619 | | 518 | 524 | 517 | 517 | | 442 | 442 | 473 | 447 | |
| 25 | 479 | 479 | 480 | 483 | | 344 | 349 | 368 | 343 | | 620 | 620 | 613 | 619 | | 517 | 521 | 518 | 517 | | 441 | 446 | 473 | 446 | |
| 26 | 480 | 480 | 480 | 483 | | 339 | 345 | 369 | 342 | | 619 | 619 | 616 | 619 | | 516 | 519 | 517 | 516 | | 443 | 447 | 461 | 446 | |
| 27 | 480 | 479 | 480 | 483 | | 339 | 339 | 369 | 342 | | 620 | 620 | 621 | 619 | | 517 | 518 | 519 | 517 | | 446 | 446 | 448 | 446 | |
| 28 | 481 | 480 | 480 | 483 | | 342 | 342 | 358 | 342 | | 619 | 619 | 620 | 618 | | 517 | 517 | 526 | 517 | | 446 | 446 | 449 | 447 | |
| 29 | 480 | | | | | 340 | | | | | 619 | | | | | 516 | | | | | 446 | | | | |
| R_0 | 718 | | | | | 502 | | | | | 768 | | | | | 577 | | | | | 672 | | | | |
| R_1 | 574 | | | | | 562 | | | | | 722 | | | | | 616 | | | | | 568 | | | | |

Ranks

TABLE 1 (Cont.)
Weight-Validities for Four Methods and Five Criteria
($N_0 = 165$)

| Criteria: Methods: | All-Univ | | | | | Math | | | | | Engl Comp | | | | | Chem | | | | | Psych | | | | |
|-----------------------|----------|-----|-----|-----|--|------|-----|-----|-----|--|-----------|-----|-----|-----|--|------|-----|-----|-----|--|-------|-----|-----|-----|--|
| | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | |
| 1 | 507 | 507 | 542 | 542 | | 266 | 322 | 393 | 393 | | 513 | 513 | 539 | 539 | | 413 | 413 | 440 | 440 | | 463 | 374 | 398 | 398 | |
| 2 | 509 | 509 | 577 | 527 | | 346 | 341 | 393 | 351 | | 546 | 551 | 531 | 537 | | 433 | 440 | 441 | 368 | | 533 | 362 | 501 | 501 | |
| 3 | 545 | 545 | 578 | 556 | | 356 | 364 | 393 | 331 | | 595 | 601 | 538 | 564 | | 477 | 450 | 432 | 366 | | 496 | 456 | 500 | 430 | |
| 4 | 553 | 533 | 588 | 553 | | 337 | 348 | 386 | 296 | | 587 | 596 | 540 | 590 | | 430 | 476 | 426 | 382 | | 510 | 501 | 508 | 395 | |
| 5 | 542 | 542 | 583 | 557 | | 318 | 329 | 389 | 302 | | 588 | 597 | 573 | 613 | | 416 | 430 | 425 | 410 | | 473 | 508 | 513 | 388 | |
| 6 | 551 | 551 | 583 | 571 | | 352 | 337 | 397 | 293 | | 578 | 605 | 596 | 599 | | 410 | 433 | 426 | 397 | | 467 | 513 | 509 | 401 | |
| 7 | 539 | 556 | 597 | 563 | | 353 | 329 | 396 | 295 | | 560 | 597 | 594 | 594 | | 400 | 422 | 425 | 381 | | 456 | 498 | 503 | 417 | |
| 8 | 548 | 564 | 610 | 559 | | 345 | 353 | 397 | 301 | | 589 | 582 | 599 | 598 | | 407 | 427 | 420 | 382 | | 471 | 483 | 519 | 412 | |
| 9 | 558 | 556 | 622 | 533 | | 323 | 347 | 363 | 276 | | 583 | 602 | 595 | 595 | | 412 | 419 | 425 | 390 | | 446 | 485 | 518 | 426 | |
| 10 | 565 | 565 | 621 | 552 | | 297 | 339 | 356 | 270 | | 576 | 607 | 594 | 601 | | 398 | 418 | 424 | 379 | | 443 | 470 | 516 | 422 | |
| 11 | 576 | 576 | 619 | 544 | | 297 | 326 | 365 | 280 | | 554 | 596 | 598 | 610 | | 398 | 417 | 423 | 380 | | 467 | 484 | 511 | 422 | |
| 12 | 563 | 563 | 601 | 536 | | 291 | 314 | 364 | 284 | | 559 | 580 | 591 | 610 | | 401 | 403 | 424 | 387 | | 468 | 468 | 529 | 418 | |
| 13 | 558 | 558 | 599 | 542 | | 287 | 314 | 366 | 272 | | 564 | 586 | 590 | 611 | | 403 | 404 | 420 | 394 | | 468 | 465 | 545 | 412 | |
| 14 | 556 | 556 | 602 | 542 | | 292 | 301 | 355 | 284 | | 560 | 582 | 612 | 613 | | 404 | 394 | 413 | 393 | | 407 | 465 | 537 | 412 | |
| 15 | 556 | 556 | 616 | 551 | | 294 | 298 | 344 | 293 | | 559 | 577 | 603 | 613 | | 405 | 400 | 420 | 385 | | 462 | 465 | 518 | 413 | |
| 16 | 562 | 562 | 600 | 560 | | 287 | 294 | 343 | 285 | | 561 | 581 | 594 | 604 | | 412 | 401 | 423 | 378 | | 452 | 460 | 514 | 421 | |
| 17 | 568 | 568 | 598 | 563 | | 282 | 300 | 343 | 282 | | 569 | 576 | 600 | 598 | | 412 | 404 | 425 | 374 | | 452 | 441 | 501 | 419 | |
| 18 | 564 | 562 | 587 | 561 | | 284 | 306 | 344 | 281 | | 576 | 575 | 602 | 591 | | 405 | 405 | 406 | 379 | | 445 | 438 | 503 | 425 | |
| 19 | 564 | 556 | 575 | 559 | | 266 | 297 | 357 | 287 | | 574 | 581 | 602 | 592 | | 405 | 405 | 406 | 383 | | 443 | 429 | 500 | 415 | |
| 20 | 564 | 555 | 573 | 557 | | 273 | 287 | 351 | 286 | | 574 | 582 | 598 | 591 | | 402 | 400 | 441 | 378 | | 421 | 431 | 502 | 404 | |
| 21 | 558 | 559 | 576 | 555 | | 281 | 285 | 359 | 283 | | 577 | 581 | 598 | 585 | | 402 | 398 | 449 | 379 | | 420 | 425 | 490 | 403 | |
| 22 | 562 | 556 | 576 | 556 | | 285 | 289 | 367 | 277 | | 577 | 581 | 598 | 589 | | 399 | 399 | 428 | 379 | | 415 | 416 | 478 | 401 | |
| 23 | 557 | 557 | 577 | 556 | | 286 | 286 | 347 | 277 | | 578 | 582 | 597 | 586 | | 397 | 397 | 439 | 380 | | 409 | 417 | 477 | 399 | |
| 24 | 559 | 559 | 579 | 555 | | 278 | 278 | 315 | 278 | | 577 | 583 | 597 | 585 | | 391 | 391 | 450 | 380 | | 407 | 409 | 468 | 398 | |
| 25 | 559 | 559 | 580 | 557 | | 278 | 278 | 315 | 278 | | 579 | 582 | 597 | 585 | | 387 | 387 | 454 | 380 | | 410 | 408 | 467 | 400 | |
| 26 | 557 | 557 | 579 | 556 | | 280 | 280 | 302 | 278 | | 582 | 583 | 592 | 583 | | 386 | 386 | 402 | 382 | | 407 | 405 | 448 | 403 | |
| 27 | 557 | 557 | 577 | 558 | | 277 | 277 | 302 | 279 | | 583 | 583 | 591 | 583 | | 387 | 387 | 394 | 380 | | 409 | 407 | 446 | 401 | |
| 28 | 556 | 556 | 558 | 558 | | 278 | 278 | 297 | 280 | | 585 | 585 | 584 | 583 | | 387 | 387 | 392 | 382 | | 408 | 408 | 445 | 400 | |
| 29 | 556 | | | | | 277 | | | | | 582 | | | | | 388 | | | | | 402 | | | | |
| R_0 | 711 | | | | | 670 | | | | | 716 | | | | | 617 | | | | | 641 | | | | |
| R_1 | 683 | | | | | 508 | | | | | 700 | | | | | 595 | | | | | 631 | | | | |

Ranks

TABLE 1 (Cont.)
Weight-Validities for Four Methods and Five Criteria
($N_0 = 120$)

| Criteria: Methods: | All-Univ | | | | | Math | | | | | Engl Comp | | | | | Chem | | | | | Psych | | | | |
|-----------------------|----------|-----|-----|-----|--|------|-----|-----|-----|--|-----------|-----|-----|-----|--|------|-----|-----|-----|--|-------|-----|-----|-----|--|
| | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | |
| 1 | 367 | 367 | 557 | 557 | | 275 | 275 | 383 | 383 | | 467 | 550 | 526 | 526 | | 369 | 369 | 425 | 425 | | 370 | 370 | 440 | 440 | |
| 2 | 448 | 448 | 504 | 507 | | 335 | 365 | 387 | 392 | | 576 | 593 | 528 | 522 | | 409 | 329 | 423 | 444 | | 456 | 454 | 492 | 492 | |
| 3 | 514 | 514 | 555 | 532 | | 293 | 340 | 385 | 362 | | 576 | 601 | 525 | 568 | | 469 | 372 | 428 | 446 | | 418 | 420 | 491 | 491 | |
| 4 | 519 | 521 | 565 | 484 | | 343 | 332 | 385 | 321 | | 589 | 588 | 570 | 583 | | 431 | 374 | 411 | 407 | | 430 | 430 | 478 | 479 | |
| 5 | 496 | 493 | 575 | 466 | | 320 | 343 | 355 | 275 | | 621 | 600 | 548 | 564 | | 403 | 398 | 413 | 394 | | 436 | 442 | 479 | 479 | |
| 6 | 459 | 515 | 576 | 485 | | 325 | 317 | 354 | 288 | | 611 | 611 | 558 | 558 | | 394 | 433 | 422 | 406 | | 437 | 446 | 483 | 440 | |
| 7 | 442 | 511 | 564 | 487 | | 315 | 321 | 350 | 289 | | 594 | 612 | 565 | 564 | | 391 | 413 | 428 | 406 | | 451 | 440 | 493 | 422 | |
| 8 | 467 | 516 | 563 | 494 | | 313 | 313 | 351 | 259 | | 591 | 604 | 565 | 577 | | 397 | 402 | 429 | 404 | | 448 | 446 | 476 | 418 | |
| 9 | 457 | 516 | 568 | 510 | | 307 | 308 | 351 | 265 | | 591 | 590 | 572 | 556 | | 408 | 402 | 421 | 381 | | 424 | 461 | 479 | 408 | |
| 10 | 442 | 513 | 566 | 520 | | 306 | 292 | 361 | 270 | | 592 | 589 | 586 | 558 | | 420 | 408 | 417 | 379 | | 397 | 412 | 488 | 410 | |
| 11 | 459 | 527 | 555 | 511 | | 291 | 271 | 371 | 269 | | 589 | 586 | 587 | 567 | | 435 | 418 | 370 | 380 | | 388 | 399 | 468 | 396 | |
| 12 | 458 | 528 | 571 | 503 | | 271 | 271 | 382 | 283 | | 579 | 583 | 580 | 563 | | 434 | 435 | 372 | 390 | | 370 | 391 | 450 | 383 | |
| 13 | 469 | 522 | 573 | 508 | | 277 | 269 | 378 | 256 | | 573 | 581 | 582 | 571 | | 432 | 435 | 375 | 401 | | 364 | 380 | 449 | 364 | |
| 14 | 477 | 509 | 572 | 503 | | 273 | 275 | 378 | 265 | | 577 | 573 | 581 | 573 | | 428 | 437 | 387 | 406 | | 367 | 373 | 450 | 329 | |
| 15 | 471 | 518 | 573 | 494 | | 271 | 272 | 348 | 277 | | 575 | 575 | 581 | 562 | | 430 | 434 | 386 | 399 | | 364 | 369 | 451 | 334 | |
| 16 | 476 | 513 | 574 | 487 | | 265 | 270 | 356 | 272 | | 570 | 577 | 595 | 553 | | 430 | 429 | 387 | 391 | | 359 | 370 | 433 | 325 | |
| 17 | 483 | 506 | 582 | 493 | | 261 | 265 | 350 | 266 | | 570 | 575 | 589 | 556 | | 437 | 427 | 394 | 399 | | 371 | 378 | 434 | 330 | |
| 18 | 494 | 505 | 534 | 494 | | 261 | 261 | 351 | 262 | | 566 | 578 | 579 | 557 | | 425 | 423 | 430 | 400 | | 376 | 376 | 449 | 345 | |
| 19 | 495 | 499 | 522 | 490 | | 266 | 261 | 342 | 265 | | 567 | 574 | 575 | 560 | | 424 | 425 | 438 | 404 | | 375 | 375 | 448 | 355 | |
| 20 | 488 | 494 | 527 | 480 | | 265 | 265 | 349 | 257 | | 569 | 578 | 575 | 566 | | 420 | 426 | 439 | 399 | | 374 | 374 | 443 | 356 | |
| 21 | 476 | 484 | 517 | 482 | | 263 | 263 | 341 | 253 | | 572 | 580 | 579 | 561 | | 419 | 426 | 446 | 401 | | 369 | 369 | 437 | 332 | |
| 22 | 478 | 472 | 496 | 473 | | 262 | 266 | 347 | 253 | | 563 | 575 | 580 | 564 | | 411 | 424 | 446 | 409 | | 369 | 367 | 414 | 350 | |
| 23 | 472 | 470 | 496 | 473 | | 262 | 266 | 359 | 257 | | 567 | 575 | 581 | 566 | | 412 | 430 | 442 | 414 | | 367 | 367 | 406 | 344 | |
| 24 | 470 | 474 | 496 | 473 | | 264 | 267 | 367 | 254 | | 569 | 570 | 571 | 568 | | 414 | 431 | 442 | 415 | | 365 | 361 | 401 | 346 | |
| 25 | 470 | 473 | 486 | 472 | | 264 | 264 | 328 | 258 | | 570 | 570 | 574 | 570 | | 415 | 424 | 445 | 414 | | 363 | 364 | 380 | 348 | |
| 26 | 471 | 476 | 478 | 472 | | 260 | 260 | 326 | 258 | | 566 | 571 | 575 | 571 | | 416 | 417 | 447 | 421 | | 356 | 363 | 380 | 350 | |
| 27 | 469 | 471 | 477 | 470 | | 258 | 258 | 297 | 258 | | 567 | 567 | 582 | 570 | | 418 | 417 | 439 | 414 | | 355 | 362 | 376 | 350 | |
| 28 | 470 | 472 | 473 | 470 | | 259 | 259 | 257 | 259 | | 567 | 567 | 586 | 569 | | 418 | 418 | 421 | 414 | | 354 | 356 | 377 | 349 | |
| 29 | 470 | | | | | 259 | | | | | 567 | | | | | 418 | | | | | 355 | | | | |
| R_0 | 764 | | | | | 670 | | | | | 788 | | | | | 629 | | | | | 692 | | | | |
| R_1 | 688 | | | | | 546 | | | | | 697 | | | | | 558 | | | | | 589 | | | | |

Ranks

TABLE 1 (Cont.)
Weight-Validities for Four Methods and Five Criteria
($N_0 = 75$)

| Criteria: Methods: | All-Univ | | | | | Math | | | | | Engl Comp | | | | | Chem | | | | | Psych | | | | |
|-----------------------|----------|-----|-----|-----|--|------|-----|-----|-----|--|-----------|-----|-----|-----|--|------|-----|-----|-----|--|-------|-----|-----|-----|--|
| | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | |
| 1 | 399 | 399 | 503 | 503 | | 363 | 363 | 403 | 403 | | 458 | 506 | 542 | 512 | | 476 | 476 | 492 | 492 | | 443 | 443 | 381 | 381 | |
| 2 | 360 | 360 | 492 | 515 | | 300 | 375 | 381 | 299 | | 581 | 545 | 542 | 580 | | 387 | 387 | 470 | 391 | | 409 | 527 | 367 | 538 | |
| 3 | 338 | 410 | 493 | 501 | | 315 | 304 | 361 | 293 | | 588 | 529 | 528 | 567 | | 304 | 304 | 479 | 372 | | 441 | 469 | 532 | 523 | |
| 4 | 314 | 388 | 492 | 511 | | 249 | 298 | 381 | 282 | | 506 | 512 | 516 | 513 | | 248 | 324 | 480 | 226 | | 425 | 441 | 511 | 533 | |
| 5 | 320 | 419 | 507 | 510 | | 299 | 291 | 336 | 244 | | 582 | 591 | 575 | 551 | | 229 | 269 | 486 | 202 | | 414 | 450 | 465 | 485 | |
| 6 | 294 | 396 | 500 | 446 | | 300 | 261 | 314 | 249 | | 588 | 593 | 561 | 579 | | 238 | 253 | 499 | 216 | | 385 | 445 | 452 | 468 | |
| 7 | 325 | 382 | 503 | 421 | | 295 | 248 | 302 | 249 | | 582 | 582 | 554 | 535 | | 259 | 247 | 493 | 215 | | 400 | 443 | 452 | 455 | |
| 8 | 317 | 365 | 497 | 373 | | 268 | 273 | 284 | 221 | | 565 | 582 | 553 | 537 | | 251 | 252 | 513 | 180 | | 359 | 390 | 445 | 437 | |
| 9 | 349 | 354 | 492 | 372 | | 263 | 270 | 279 | 211 | | 561 | 575 | 553 | 529 | | 259 | 227 | 514 | 211 | | 353 | 371 | 446 | 420 | |
| 10 | 347 | 363 | 492 | 377 | | 272 | 264 | 284 | 213 | | 557 | 575 | 541 | 528 | | 229 | 222 | 436 | 206 | | 356 | 363 | 429 | 414 | |
| 11 | 370 | 359 | 486 | 367 | | 267 | 257 | 257 | 202 | | 545 | 568 | 537 | 500 | | 252 | 237 | 429 | 161 | | 341 | 360 | 418 | 386 | |
| 12 | 354 | 341 | 495 | 374 | | 258 | 250 | 280 | 220 | | 527 | 569 | 530 | 483 | | 247 | 248 | 450 | 179 | | 312 | 383 | 449 | 384 | |
| 13 | 352 | 335 | 493 | 371 | | 254 | 243 | 280 | 231 | | 519 | 568 | 567 | 482 | | 241 | 258 | 426 | 156 | | 291 | 376 | 451 | 386 | |
| 14 | 324 | 340 | 489 | 366 | | 255 | 239 | 278 | 218 | | 492 | 561 | 576 | 480 | | 208 | 261 | 430 | 169 | | 320 | 337 | 435 | 369 | |
| 15 | 326 | 316 | 479 | 358 | | 246 | 240 | 213 | 207 | | 500 | 564 | 572 | 471 | | 219 | 250 | 390 | 168 | | 320 | 333 | 419 | 370 | |
| 16 | 325 | 332 | 468 | 353 | | 231 | 239 | 224 | 222 | | 507 | 563 | 557 | 463 | | 212 | 245 | 305 | 178 | | 316 | 340 | 434 | 332 | |
| 17 | 333 | 333 | 441 | 355 | | 230 | 219 | 226 | 215 | | 482 | 543 | 551 | 459 | | 219 | 253 | 307 | 190 | | 328 | 326 | 433 | 315 | |
| 18 | 335 | 335 | 445 | 353 | | 224 | 225 | 233 | 201 | | 470 | 541 | 558 | 469 | | 225 | 240 | 316 | 212 | | 335 | 330 | 418 | 317 | |
| 19 | 344 | 344 | 441 | 348 | | 222 | 213 | 239 | 196 | | 462 | 520 | 558 | 475 | | 215 | 223 | 265 | 194 | | 337 | 328 | 411 | 310 | |
| 20 | 336 | 336 | 427 | 338 | | 223 | 212 | 248 | 198 | | 458 | 496 | 556 | 466 | | 211 | 211 | 283 | 201 | | 339 | 342 | 423 | 301 | |
| 21 | 325 | 325 | 378 | 338 | | 222 | 209 | 242 | 203 | | 471 | 483 | 567 | 465 | | 215 | 202 | 272 | 190 | | 328 | 332 | 418 | 315 | |
| 22 | 325 | 325 | 371 | 343 | | 222 | 211 | 245 | 199 | | 475 | 476 | 571 | 464 | | 215 | 201 | 292 | 203 | | 332 | 335 | 403 | 325 | |
| 23 | 320 | 320 | 371 | 336 | | 221 | 211 | 221 | 204 | | 477 | 474 | 535 | 468 | | 207 | 205 | 252 | 199 | | 323 | 333 | 396 | 320 | |
| 24 | 322 | 322 | 360 | 336 | | 216 | 212 | 201 | 206 | | 467 | 480 | 532 | 467 | | 204 | 205 | 233 | 200 | | 320 | 326 | 370 | 315 | |
| 25 | 320 | 320 | 362 | 330 | | 212 | 211 | 197 | 208 | | 471 | 469 | 522 | 472 | | 205 | 206 | 219 | 201 | | 319 | 321 | 371 | 317 | |
| 26 | 322 | 322 | 358 | 329 | | 210 | 212 | 197 | 206 | | 471 | 471 | 511 | 476 | | 206 | 204 | 219 | 205 | | 319 | 320 | 353 | 317 | |
| 27 | 324 | 324 | 331 | 326 | | 210 | 205 | 200 | 205 | | 470 | 470 | 514 | 478 | | 203 | 205 | 225 | 204 | | 319 | 320 | 351 | 318 | |
| 28 | 325 | 325 | 324 | 325 | | 208 | 208 | 204 | 205 | | 470 | 471 | 470 | 476 | | 203 | 203 | 204 | 206 | | 319 | 319 | 354 | 319 | |
| 29 | 325 | | | | | 208 | | | | | 475 | | | | | 204 | | | | | 319 | | | | |
| R_0 | 655 | | | | | 755 | | | | | 748 | | | | | 760 | | | | | 790 | | | | |
| R_1 | 572 | | | | | 528 | | | | | 716 | | | | | 620 | | | | | 609 | | | | |

Ranks

TABLE 1 (Cont.)
Weight-Validities for Four Methods and Five Criteria
($N_0 = 30$)

| Criteria: Methods: | All-Univ | | | | | Math | | | | | Engl Comp | | | | | Chem | | | | | Psych | | | | |
|-----------------------|----------|-----|-----|-----|--|------|-----|-----|------|--|-----------|-----|-----|------|--|------|-----|-----|------|--|-------|-----|-----|------|--|
| | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | |
| 1 | 461 | 428 | 481 | 481 | | 347 | 347 | 397 | 397 | | 440 | 501 | 577 | 577 | | 365 | 292 | 431 | 431 | | 298 | 298 | 464 | 080 | |
| 2 | 366 | 400 | 478 | 361 | | 361 | 316 | 424 | 319 | | 543 | 451 | 534 | 534 | | 375 | 267 | 432 | 303 | | 285 | 285 | 406 | 406 | |
| 3 | 416 | 333 | 531 | 432 | | 398 | 272 | 423 | 156 | | 519 | 486 | 502 | 502 | | 354 | 288 | 440 | 010 | | 317 | 347 | 430 | 385 | |
| 4 | 372 | 390 | 513 | 022 | | 340 | 267 | 413 | 174 | | 556 | 527 | 404 | 466 | | 347 | 349 | 437 | 022 | | 343 | 301 | 419 | 403 | |
| 5 | 393 | 381 | 472 | 022 | | 270 | 275 | 433 | 212 | | 518 | 483 | 563 | 509 | | 331 | 364 | 431 | 018 | | 390 | 291 | 441 | 303 | |
| 6 | 380 | 398 | 467 | 022 | | 168 | 247 | 370 | -006 | | 520 | 514 | 563 | 520 | | 329 | 426 | 421 | 014 | | 408 | 281 | 451 | 142 | |
| 7 | 354 | 395 | 472 | 022 | | 179 | 251 | 364 | 006 | | 477 | 514 | 575 | -004 | | 317 | 365 | 426 | 096 | | 357 | 312 | 426 | 152 | |
| 8 | 356 | 365 | 467 | 022 | | 149 | 265 | 367 | 006 | | 482 | 470 | 574 | -004 | | 308 | 350 | 430 | 086 | | 313 | 289 | 422 | 143 | |
| 9 | 332 | 353 | 467 | 015 | | 130 | 283 | 347 | 076 | | 484 | 471 | 590 | -004 | | 317 | 332 | 421 | 095 | | 274 | 248 | 439 | 147 | |
| 10 | 309 | 377 | 449 | 018 | | 112 | 273 | 346 | 089 | | 489 | 450 | 597 | -004 | | 298 | 316 | 434 | -018 | | 209 | 232 | 405 | 142 | |
| 11 | 328 | 365 | 454 | 018 | | 106 | 271 | 317 | 092 | | 462 | 460 | 598 | -003 | | 293 | 304 | 434 | -018 | | 186 | 259 | 389 | 137 | |
| 12 | 324 | 338 | 451 | 018 | | 101 | 278 | 318 | 081 | | 397 | 468 | 598 | -002 | | 284 | 271 | 430 | -019 | | 174 | 248 | 386 | 145 | |
| 13 | 342 | 324 | 414 | 018 | | 098 | 256 | 318 | 088 | | 389 | 453 | 554 | -004 | | 289 | 245 | 408 | -019 | | 174 | 259 | 352 | 138 | |
| 14 | 325 | 317 | 410 | 018 | | 064 | 234 | 295 | -002 | | 374 | 418 | 553 | -004 | | 285 | 208 | 397 | -018 | | 181 | 259 | 343 | 130 | |
| 15 | 328 | 322 | 391 | 017 | | 056 | 229 | 278 | 000 | | 347 | 395 | 530 | -005 | | 282 | 177 | 397 | -018 | | 182 | 269 | 342 | 124 | |
| 16 | 301 | 338 | 379 | 019 | | 016 | 231 | 288 | -006 | | 304 | 360 | 521 | -004 | | 220 | 160 | 361 | -016 | | 162 | 256 | 349 | 126 | |
| 17 | 289 | 354 | 345 | 019 | | 031 | 194 | 287 | -008 | | 228 | 363 | 497 | -003 | | 129 | 137 | 334 | -015 | | 136 | 226 | 341 | -029 | |
| 18 | 280 | 370 | 310 | 019 | | 021 | 167 | 287 | -009 | | 195 | 334 | 488 | -003 | | 055 | 131 | 334 | -017 | | 083 | 181 | 311 | -024 | |
| 19 | 223 | 387 | 249 | 021 | | -011 | 148 | 177 | -006 | | 147 | 284 | 483 | -003 | | 011 | 138 | 295 | -015 | | 068 | 190 | 333 | -024 | |
| 20 | 169 | 336 | 247 | 021 | | -024 | 086 | 202 | -007 | | 141 | 267 | 477 | -002 | | -015 | 135 | 298 | -016 | | 011 | 172 | 266 | -025 | |
| 21 | 137 | 315 | 349 | 021 | | -034 | 088 | 177 | -009 | | 137 | 261 | 476 | -002 | | -018 | 125 | 301 | -014 | | 010 | 170 | 235 | -024 | |
| 22 | 091 | 312 | 379 | 024 | | -033 | 086 | 219 | -009 | | 140 | 257 | 471 | -002 | | -017 | 063 | 301 | -013 | | 005 | 158 | 257 | -025 | |
| 23 | 076 | 334 | 390 | 024 | | -033 | 079 | 210 | -009 | | 113 | 181 | 454 | -002 | | -021 | 056 | 268 | -015 | | -002 | 156 | 257 | -026 | |
| 24 | 062 | 321 | 373 | 021 | | -028 | 076 | 149 | -011 | | 078 | 173 | 423 | -001 | | -029 | 048 | 211 | -014 | | -036 | 155 | 224 | -029 | |
| 25 | 057 | 170 | 318 | 021 | | -022 | 069 | 236 | -012 | | 050 | 157 | 403 | -001 | | -028 | 049 | 215 | -014 | | -025 | 155 | 166 | -028 | |
| 26 | 031 | 075 | 311 | 024 | | -013 | 044 | 204 | -011 | | 036 | 142 | 333 | -002 | | -027 | 064 | 212 | -014 | | -013 | 144 | 166 | -027 | |
| 27 | 019 | 074 | 302 | 024 | | -011 | 031 | 180 | -011 | | 002 | 135 | 218 | -002 | | -014 | 110 | 025 | -014 | | -013 | 139 | 134 | -023 | |
| 28 | 024 | 076 | 073 | 021 | | -011 | 018 | 042 | -012 | | 000 | 055 | 214 | -002 | | -023 | 029 | 104 | -013 | | -014 | 048 | 143 | -024 | |
| 29 | 021 | | | | | -021 | | | | | -002 | | | | | -013 | | | | | -024 | | | | |
| R_0 | 999 | | | | | 999 | | | | | 999 | | | | | 999 | | | | | 999 | | | | |
| R_1 | 663 | | | | | 581 | | | | | 693 | | | | | 565 | | | | | 615 | | | | |

Ranks

six original-sample sizes used, ranging from 255 down to 30 cases. This size is denoted by N_0 . In each instance, the new sample contained 252 cases. An original sample and a new sample were independently drawn for each size and each criterion, for a total of 30 original samples and 30 new samples. Since for rank 29, all methods are equivalent (aside from rounding error), the corresponding weight-validity is listed only under method 1. The full-rank (rank 29) multiple correlations for each sample are also listed under method 1, the subscripts 0 and 1 denoting the original and new samples, respectively.

Although the weight-validities using method 4 were computed on the basis of the data given above, they are not presented. For all ranks, criteria, and sample sizes, these weight-validities were substantially lower than those for any other method or for the full-rank weights. They were frequently negative, rarely greater than .10, and virtually always less than half as large as the weight-validities obtained by any of the other methods. Our expectation that the method of smallest principal-axes factors would give less accurate predictions than the other methods is thus unequivocally confirmed.

To assist in comparing the other four reduced-rank methods, Table 2 was prepared from Table 1. For each original-sample size and each criterion, two comparisons are made. In each of the first five columns, the number of ranks for which each method was superior to the other three methods is given. In making the counts, ties were divided equally among the methods sharing the high value for a particular rank. In each of the second five columns of Table 2, the number of ranks for which a particular method was superior to the full-rank weights is given. When for a particular rank a method had the same weight-validity as the full-rank weights, the count was increased by one half.

Of the four methods, the method of largest principal-axes factors most often gave the highest weight-validities in 26 of the 30 samples. This trend was most marked when the weights were computed on smaller samples, particularly samples of size 30. The only exceptions occurred for samples of 210 and 255 cases. The superiority of method 3 was most pronounced for Psychology and Mathematics and less clear-cut for English Composition and Chemistry. Method 3 was also more often superior to the full-rank weights than were the other methods. Thus it appears that our expectation as to the superiority of method 3 is also confirmed, but with the qualification that, for larger samples and for certain criterion variables, one or more of the other methods may be preferable.

Another possible basis of comparison would be the number of samples for which a particular method gave the highest weight-validity for any rank. Of the 30 samples, method 3 gave the highest validity in 12.5, method 5 in 8.5, method 1 in 5, and method 2 in 4 samples. The comparisons of Table 2 would appear to be more meaningful than this comparison, however, since

TABLE 2
Comparisons Between Four Reduced-Rank Methods With Respect to Weight-Validities for Five Criteria

| Sample Size | Methods | Number of ranks for which weight-validity is higher than for other methods | | | | Number of ranks for which weight-validity is higher than full-rank method | | | | |
|-------------|---------|--|-------|-------|------|---|----------|------|------|------|
| | | All-Univ | Math | Engl | Chem | Psych | All-Univ | Math | Engl | Chem |
| 255 | 1 | 0. | 2.75 | 5.17 | 2. | 6. | 5. | 13. | 9.5 | 16.5 |
| | 2 | 0. | .75 | 3.33 | 5. | 2. | 6.5 | 13.5 | 8. | 15.5 |
| | 3 | 24.5 | 19.75 | 13.83 | 6. | 19.5 | 28. | 28. | 18. | 11.5 |
| | 5 | 3.5 | 4.75 | 5.67 | 15. | .5 | 26. | 24.5 | 17. | 21. |
| | 1 | .5 | 0. | 6.17 | 8. | 0. | 5. | 26. | 19. | 22.5 |
| 210 | 2 | .5 | 0. | 9.67 | 17. | 0. | 13. | 26. | 20. | 11. |
| | 3 | 11.5 | 19.5 | 8. | 2.5 | 27. | 24.5 | 28. | 14.5 | 8. |
| | 5 | 15.5 | 8.5 | 4.17 | .5 | 1. | 26. | 27. | 4. | 12. |
| | 1 | 0. | 0. | 0. | 1. | 3. | 18.5 | 24.5 | 7. | 24. |
| | 2 | 0. | 0. | 7. | 4. | 1. | 17.5 | 27.5 | 15. | 24. |
| 165 | 3 | 27. | 27.5 | 13.5 | 22.5 | 24. | 27. | 28. | 23. | 27. |
| | 5 | 1. | .5 | 7.5 | .5 | 0. | 14. | 24. | 25. | 6. |
| | 1 | 0. | .33 | 5.5 | 9. | 0. | 15.5 | 26.5 | 22. | 15. |
| | 2 | 0. | .33 | 9.5 | 5. | 0. | 25.5 | 26.5 | 26. | 16. |
| | 3 | 26.5 | 25.5 | 13. | 15.5 | 25. | 28. | 27. | 21. | 18. |
| 120 | 5 | 1.5 | 1.83 | 0. | 1.5 | 3. | 26. | 18. | 10.5 | 4. |
| | 1 | .33 | 7. | 3.5 | 0. | .5 | 15.5 | 27.5 | 17.5 | 25.5 |
| | 2 | .33 | 2.5 | 9.5 | 0. | .5 | 20.5 | 26.5 | 23. | 24.5 |
| | 3 | 22.5 | 18. | 13.5 | 26.5 | 22. | 27. | 27. | 27. | 27.5 |
| | 5 | 4.83 | .5 | 1.5 | 1.5 | 5. | 27.5 | 15.5 | 17.5 | 18.5 |
| 75 | 1 | 0. | 0. | 3. | 0. | 0. | 26.5 | 23. | 28. | 26. |
| | 2 | 5. | 0. | 0. | 2. | 1. | 28. | 28. | 28. | 28. |
| | 3 | 22.5 | 27.5 | 24.5 | 25.5 | 26.5 | 28. | 28. | 28. | 28. |
| | 5 | .5 | .5 | .5 | .5 | .5 | 13. | 27. | 12. | 10. |
| | 1 | .5 | .5 | .5 | .5 | .5 | 13. | 27. | 12. | 10. |

TABLE 3
Total Squared Errors of Prediction and Weight-Validities for Four Methods and a Single Criterion

| Methods | First New Sample | | | | | Second New Sample | | | | | Total Errors | | | | |
|---------|-------------------|-----|-----|-----|-----|-------------------|-----|-----|-----|-----|--------------|-----|-----|-----|-----|
| | Weight-Validities | | | | | Weight-Validities | | | | | Total Errors | | | | |
| | 1 | 2 | 3 | 5 | 788 | 746 | 663 | 663 | 663 | 385 | 385 | 488 | 488 | 860 | 865 |
| 1 | 461 | 504 | 582 | 582 | 742 | 691 | 617 | 617 | 617 | 407 | 422 | 487 | 459 | 852 | 831 |
| 2 | 509 | 556 | 596 | 570 | 666 | 673 | 617 | 661 | 661 | 462 | 452 | 487 | 456 | 802 | 804 |
| 3 | 579 | 572 | 596 | 583 | 650 | 646 | 643 | 653 | 653 | 467 | 462 | 491 | 469 | 797 | 800 |
| 4 | 592 | 595 | 599 | 589 | 644 | 625 | 638 | 649 | 649 | 480 | 478 | 499 | 477 | 789 | 785 |
| 5 | 597 | 612 | 603 | 593 | 628 | 626 | 633 | 640 | 640 | 493 | 485 | 503 | 487 | 773 | 781 |
| 6 | 610 | 612 | 608 | 600 | 620 | 619 | 619 | 612 | 612 | 493 | 494 | 513 | 480 | 773 | 770 |
| 7 | 616 | 617 | 619 | 599 | 621 | 630 | 617 | 634 | 634 | 490 | 476 | 513 | 493 | 781 | 795 |
| 8 | 616 | 609 | 620 | 605 | 621 | 630 | 617 | 634 | 634 | 484 | 471 | 515 | 491 | 790 | 801 |
| 9 | 616 | 608 | 620 | 604 | 621 | 631 | 617 | 635 | 635 | 486 | 469 | 516 | 485 | 786 | 807 |
| 10 | 622 | 608 | 620 | 600 | 613 | 631 | 618 | 640 | 640 | 471 | 471 | 514 | 477 | 807 | 807 |
| 11 | 616 | 616 | 613 | 596 | 621 | 621 | 625 | 616 | 616 | 472 | 472 | 514 | 478 | 806 | 806 |
| 12 | 607 | 607 | 613 | 591 | 634 | 631 | 625 | 618 | 618 | 472 | 472 | 514 | 477 | 808 | 808 |
| 13 | 607 | 607 | 613 | 588 | 634 | 634 | 625 | 635 | 635 | 470 | 470 | 514 | 477 | 807 | 814 |
| 14 | 605 | 601 | 614 | 593 | 636 | 642 | 624 | 650 | 650 | 470 | 465 | 511 | 485 | 810 | 813 |
| 15 | 606 | 601 | 613 | 588 | 635 | 642 | 625 | 657 | 657 | 469 | 466 | 507 | 481 | 816 | 816 |
| 16 | 601 | 601 | 617 | 590 | 642 | 642 | 620 | 654 | 654 | 464 | 464 | 518 | 475 | 816 | 816 |
| 17 | 606 | 606 | 617 | 595 | 635 | 635 | 619 | 648 | 648 | 464 | 464 | 514 | 474 | 816 | 816 |
| 18 | 608 | 608 | 624 | 595 | 633 | 633 | 611 | 648 | 648 | 468 | 468 | 525 | 471 | 812 | 812 |
| 19 | 609 | 609 | 622 | 602 | 632 | 632 | 613 | 640 | 640 | 467 | 467 | 526 | 468 | 813 | 813 |
| 20 | 612 | 612 | 629 | 607 | 628 | 628 | 604 | 633 | 633 | 470 | 470 | 521 | 471 | 809 | 809 |
| 21 | 611 | 611 | 614 | 611 | 628 | 628 | 623 | 628 | 628 | 468 | 468 | 491 | 474 | 812 | 812 |
| 22 | 612 | 612 | 619 | 611 | 629 | 629 | 617 | 629 | 629 | 472 | 472 | 500 | 472 | 807 | 807 |
| 23 | 611 | 611 | 617 | 610 | 628 | 628 | 620 | 629 | 629 | 472 | 472 | 496 | 472 | 806 | 806 |
| 24 | 611 | 611 | 621 | 610 | 628 | 628 | 616 | 630 | 630 | 473 | 473 | 492 | 470 | 805 | 805 |
| 25 | 613 | 613 | 615 | 612 | 626 | 626 | 623 | 628 | 628 | 472 | 472 | 488 | 471 | 807 | 807 |
| 26 | 613 | 613 | 615 | 612 | 626 | 626 | 624 | 627 | 627 | 472 | 472 | 486 | 472 | 807 | 807 |
| 27 | 613 | 613 | 610 | 612 | 626 | 626 | 630 | 627 | 627 | 472 | 472 | 478 | 472 | 806 | 806 |
| 28 | 612 | 612 | 608 | 612 | 627 | 627 | 633 | 627 | 627 | 472 | 472 | 472 | 472 | 806 | 806 |
| 29 | 613 | | | | 626 | | | | | 472 | | | | 806 | |
| | $N_0 = 435$ | | | | | $R_0 = 626$ | | | | | $R_1 = 684$ | | | | |
| | | | | | | | | | | | $R_2 = 582$ | | | | |

Decimal point preceding each entry has been omitted.

TABLE 3 (Cont.)
Total Squared Errors of Prediction and Weight-Validities for Four Methods and a Single Criterion

| Methods | First New Sample | | | | | Total Errors | | | | | Second New Sample | | | | | Total Errors | | | | |
|---------|-------------------|-----|-----|-----|-----|--------------|-----|-----|-----|-----|-------------------|-----|-----|-----|-----|--------------|-----|-----|-----|-----|
| | Weight-Validities | | | | | 1 | 2 | 3 | 5 | 747 | Weight-Validities | | | | | 1 | 2 | 3 | 5 | 723 |
| | 1 | 2 | 3 | 5 | 747 | | | | | | 1 | 2 | 3 | 5 | 747 | | | | | |
| 1 | 383 | 408 | 511 | 511 | 859 | 843 | 747 | 747 | 747 | 377 | 406 | 531 | 531 | 531 | 866 | 845 | 723 | 723 | 723 | 723 |
| 2 | 402 | 446 | 524 | 507 | 872 | 820 | 732 | 759 | 759 | 426 | 450 | 535 | 523 | 523 | 837 | 813 | 718 | 736 | 736 | 736 |
| 3 | 477 | 440 | 518 | 530 | 791 | 839 | 741 | 731 | 731 | 480 | 466 | 530 | 532 | 532 | 780 | 799 | 724 | 727 | 727 | 727 |
| 4 | 489 | 489 | 516 | 526 | 780 | 780 | 753 | 740 | 740 | 496 | 496 | 524 | 514 | 514 | 764 | 764 | 735 | 754 | 754 | 754 |
| 5 | 487 | 487 | 523 | 511 | 788 | 788 | 742 | 759 | 759 | 491 | 491 | 530 | 505 | 505 | 773 | 773 | 727 | 767 | 767 | 767 |
| 6 | 521 | 521 | 527 | 509 | 715 | 715 | 735 | 760 | 760 | 515 | 515 | 536 | 508 | 508 | 746 | 746 | 720 | 763 | 763 | 763 |
| 7 | 511 | 511 | 530 | 504 | 759 | 759 | 731 | 770 | 770 | 512 | 512 | 536 | 504 | 504 | 750 | 750 | 721 | 768 | 768 | 768 |
| 8 | 514 | 514 | 534 | 505 | 754 | 754 | 726 | 770 | 770 | 516 | 522 | 535 | 505 | 505 | 747 | 741 | 722 | 768 | 768 | 768 |
| 9 | 514 | 528 | 535 | 518 | 754 | 737 | 726 | 756 | 756 | 513 | 520 | 535 | 510 | 510 | 752 | 745 | 722 | 761 | 761 | 761 |
| 10 | 505 | 529 | 537 | 524 | 769 | 737 | 724 | 746 | 746 | 508 | 512 | 532 | 515 | 515 | 762 | 757 | 727 | 753 | 753 | 753 |
| 11 | 518 | 528 | 536 | 520 | 754 | 738 | 727 | 754 | 754 | 509 | 504 | 531 | 504 | 504 | 764 | 766 | 730 | 768 | 768 | 768 |
| 12 | 531 | 538 | 534 | 514 | 737 | 726 | 735 | 759 | 759 | 506 | 503 | 513 | 505 | 505 | 771 | 770 | 756 | 767 | 767 | 767 |
| 13 | 534 | 538 | 531 | 517 | 733 | 727 | 738 | 756 | 756 | 500 | 507 | 506 | 502 | 502 | 780 | 766 | 768 | 772 | 772 | 772 |
| 14 | 534 | 542 | 530 | 530 | 734 | 721 | 739 | 740 | 740 | 505 | 513 | 505 | 510 | 510 | 774 | 759 | 768 | 761 | 761 | 761 |
| 15 | 529 | 536 | 549 | 525 | 741 | 729 | 716 | 746 | 746 | 506 | 512 | 516 | 509 | 509 | 773 | 759 | 759 | 764 | 764 | 764 |
| 16 | 530 | 532 | 546 | 524 | 739 | 737 | 722 | 747 | 747 | 511 | 512 | 507 | 502 | 502 | 766 | 761 | 771 | 776 | 776 | 776 |
| 17 | 536 | 531 | 543 | 527 | 732 | 739 | 726 | 742 | 742 | 519 | 508 | 510 | 508 | 508 | 756 | 771 | 766 | 768 | 768 | 768 |
| 18 | 538 | 524 | 541 | 530 | 731 | 747 | 728 | 738 | 738 | 514 | 502 | 513 | 507 | 507 | 763 | 780 | 762 | 770 | 770 | 770 |
| 19 | 533 | 527 | 541 | 535 | 738 | 714 | 728 | 733 | 733 | 509 | 504 | 512 | 506 | 506 | 770 | 777 | 763 | 773 | 773 | 773 |
| 20 | 533 | 527 | 553 | 536 | 737 | 745 | 712 | 733 | 733 | 506 | 500 | 519 | 507 | 507 | 774 | 783 | 752 | 773 | 773 | 773 |
| 21 | 535 | 530 | 547 | 533 | 736 | 739 | 719 | 738 | 738 | 506 | 507 | 521 | 501 | 501 | 775 | 774 | 750 | 783 | 783 | 783 |
| 22 | 533 | 533 | 550 | 532 | 737 | 737 | 716 | 739 | 739 | 503 | 503 | 518 | 495 | 495 | 779 | 779 | 756 | 792 | 792 | 792 |
| 23 | 535 | 535 | 552 | 530 | 735 | 735 | 712 | 740 | 740 | 503 | 503 | 519 | 497 | 497 | 781 | 781 | 755 | 788 | 788 | 788 |
| 24 | 534 | 534 | 548 | 531 | 735 | 735 | 717 | 740 | 740 | 502 | 502 | 518 | 501 | 501 | 783 | 783 | 757 | 784 | 784 | 784 |
| 25 | 534 | 534 | 535 | 534 | 735 | 735 | 732 | 736 | 736 | 503 | 503 | 509 | 502 | 502 | 781 | 781 | 769 | 782 | 782 | 782 |
| 26 | 535 | 535 | 535 | 534 | 733 | 733 | 732 | 736 | 736 | 504 | 504 | 509 | 502 | 502 | 780 | 780 | 769 | 782 | 782 | 782 |
| 27 | 534 | 534 | 535 | 533 | 735 | 735 | 732 | 737 | 737 | 502 | 502 | 503 | 502 | 502 | 782 | 782 | 778 | 782 | 782 | 782 |
| 28 | 533 | 533 | 532 | 532 | 736 | 736 | 737 | 737 | 737 | 502 | 502 | 497 | 501 | 501 | 782 | 782 | 788 | 788 | 788 | 788 |
| 29 | 533 | | | | 736 | | | | | 502 | | | | | | | | | | |
| | $N_0 = 315$ | | | | | $R_0 = 676$ | | | | | $R_1 = 649$ | | | | | $R_2 = 630$ | | | | |

Ranks

TABLE 3 (Cont.)
Total Squared Errors of Prediction and Wave-Validities for Four Methods and a Single Criterion

| Methods | First New Sample | | | | | Second New Sample | | | | | Total Errors | | | | |
|---------|-------------------|-----|-----|-----|--|-------------------|-----|-----|-----|--|--------------|-----|-----|-----|--|
| | Weight-Validities | | | | | Weight-Validities | | | | | Total Errors | | | | |
| | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | |
| 1 | 457 | 457 | 561 | 561 | | 794 | 667 | 645 | 665 | | 383 | 473 | 473 | 473 | |
| 2 | 503 | 526 | 592 | 592 | | 749 | 723 | 650 | 650 | | 422 | 422 | 443 | 443 | |
| 3 | 575 | 561 | 590 | 587 | | 670 | 686 | 652 | 655 | | 432 | 420 | 443 | 445 | |
| 4 | 565 | 586 | 590 | 600 | | 684 | 657 | 652 | 640 | | 442 | 439 | 443 | 450 | |
| 5 | 574 | 574 | 507 | 580 | | 673 | 672 | 644 | 665 | | 454 | 446 | 446 | 442 | |
| 6 | 574 | 579 | 596 | 580 | | 673 | 667 | 645 | 665 | | 463 | 456 | 439 | 433 | |
| 7 | 568 | 574 | 612 | 572 | | 682 | 673 | 627 | 676 | | 462 | 463 | 446 | 437 | |
| 8 | 569 | 570 | 611 | 571 | | 680 | 679 | 627 | 677 | | 463 | 463 | 446 | 442 | |
| 9 | 561 | 581 | 612 | 581 | | 692 | 666 | 626 | 661 | | 456 | 459 | 446 | 449 | |
| 10 | 561 | 579 | 612 | 581 | | 692 | 670 | 626 | 662 | | 456 | 464 | 447 | 450 | |
| 11 | 565 | 580 | 614 | 592 | | 686 | 669 | 624 | 652 | | 456 | 464 | 449 | 448 | |
| 12 | 566 | 580 | 608 | 582 | | 685 | 668 | 631 | 665 | | 449 | 463 | 441 | 452 | |
| 13 | 569 | 574 | 607 | 575 | | 682 | 675 | 633 | 674 | | 450 | 463 | 436 | 458 | |
| 14 | 574 | 569 | 616 | 584 | | 675 | 683 | 622 | 663 | | 449 | 459 | 435 | 462 | |
| 15 | 580 | 570 | 612 | 581 | | 667 | 681 | 625 | 667 | | 456 | 458 | 438 | 465 | |
| 16 | 577 | 571 | 608 | 579 | | 672 | 681 | 630 | 669 | | 452 | 462 | 441 | 467 | |
| 17 | 580 | 571 | 608 | 578 | | 667 | 680 | 631 | 672 | | 447 | 457 | 435 | 465 | |
| 18 | 582 | 573 | 620 | 574 | | 665 | 677 | 616 | 676 | | 447 | 456 | 442 | 459 | |
| 19 | 578 | 575 | 602 | 573 | | 671 | 674 | 638 | 676 | | 447 | 456 | 434 | 453 | |
| 20 | 580 | 579 | 600 | 571 | | 668 | 669 | 640 | 679 | | 445 | 454 | 435 | 453 | |
| 21 | 582 | 575 | 599 | 575 | | 666 | 674 | 642 | 674 | | 448 | 449 | 435 | 451 | |
| 22 | 578 | 578 | 600 | 576 | | 671 | 670 | 640 | 673 | | 448 | 452 | 443 | 451 | |
| 23 | 579 | 578 | 589 | 575 | | 670 | 670 | 654 | 674 | | 448 | 453 | 449 | 452 | |
| 24 | 578 | 580 | 591 | 577 | | 671 | 667 | 652 | 672 | | 449 | 447 | 449 | 454 | |
| 25 | 577 | 577 | 581 | 574 | | 672 | 672 | 667 | 675 | | 448 | 447 | 452 | 454 | |
| 26 | 575 | 578 | 573 | 573 | | 674 | 670 | 676 | 677 | | 448 | 448 | 457 | 453 | |
| 27 | 575 | 577 | 572 | 573 | | 674 | 672 | 678 | 677 | | 448 | 448 | 457 | 453 | |
| 28 | 576 | 575 | 569 | 573 | | 673 | 674 | 682 | 677 | | 448 | 448 | 456 | 454 | |
| 29 | 576 | | | | | 673 | | | | | 448 | | | | |
| | $N_s = 255$ | | | | | $R_0 = 646$ | | | | | $R_1 = 717$ | | | | |
| | | | | | | | | | | | $R_2 = 563$ | | | | |

Ranks

[illegible]

| Ranks | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| | 380 | 418 | 474 | 492 | 529 | 517 | 517 | 512 | 496 | 462 | 457 | 439 | 432 | 423 | 427 | 424 | 430 | 436 | 434 | 433 | 438 | 444 | 447 | 449 | 444 | 445 | 443 | 443 | 443 |
| | 380 | 418 | 474 | 492 | 529 | 517 | 514 | 498 | 473 | 467 | 465 | 473 | 452 | 433 | 433 | 430 | 427 | 426 | 426 | 431 | 425 | 427 | 428 | 426 | 428 | 433 | 432 | 442 | 443 |
| | 546 | 536 | 536 | 531 | 534 | 541 | 542 | 521 | 512 | 516 | 509 | 509 | 509 | 520 | 522 | 506 | 506 | 479 | 479 | 484 | 485 | 475 | 478 | 482 | 466 | 462 | 448 | 441 | 442 |
| | 865 | 865 | 806 | 792 | 749 | 773 | 777 | 804 | 807 | 858 | 865 | 894 | 910 | 928 | 926 | 935 | 931 | 920 | 916 | 925 | 920 | 910 | 905 | 902 | 912 | 910 | 913 | 914 | 914 |
| | 865 | 843 | 806 | 792 | 749 | 773 | 777 | 804 | 807 | 858 | 852 | 843 | 805 | 929 | 925 | 932 | 940 | 942 | 944 | 934 | 945 | 941 | 941 | 943 | 935 | 926 | 931 | 916 | 914 |
| | 703 | 718 | 718 | 720 | 721 | 713 | 712 | 743 | 754 | 750 | 759 | 759 | 759 | 752 | 749 | 775 | 789 | 816 | 816 | 817 | 816 | 835 | 838 | 833 | 863 | 870 | 904 | 917 | 915 |
| | 703 | 720 | 729 | 741 | 767 | 759 | 748 | 766 | 807 | 808 | 839 | 831 | 852 | 881 | 898 | 898 | 909 | 915 | 916 | 917 | 916 | 918 | 915 | 917 | 918 | 917 | 914 | 915 | 915 |
| | 451 | 481 | 467 | 467 | 504 | 482 | 471 | 483 | 467 | 451 | 435 | 420 | 431 | 435 | 429 | 435 | 437 | 448 | 449 | 442 | 442 | 446 | 443 | 443 | 444 | 441 | 441 | 441 | 441 |
| | 451 | 481 | 467 | 467 | 504 | 482 | 471 | 483 | 467 | 451 | 435 | 420 | 431 | 435 | 429 | 435 | 437 | 448 | 449 | 442 | 442 | 446 | 443 | 443 | 444 | 441 | 441 | 441 | 441 |
| | 514 | 514 | 545 | 545 | 514 | 527 | 533 | 501 | 505 | 504 | 490 | 490 | 490 | 503 | 502 | 480 | 487 | 479 | 473 | 474 | 473 | 480 | 454 | 450 | 432 | 428 | 434 | 440 | 441 |
| | 514 | 519 | 545 | 545 | 514 | 527 | 533 | 501 | 505 | 504 | 490 | 490 | 490 | 503 | 502 | 480 | 487 | 479 | 473 | 474 | 473 | 480 | 454 | 450 | 432 | 428 | 434 | 440 | 441 |
| | 797 | 774 | 811 | 788 | 776 | 809 | 807 | 820 | 845 | 875 | 895 | 927 | 917 | 919 | 931 | 935 | 936 | 918 | 901 | 901 | 915 | 906 | 909 | 908 | 910 | 915 | 912 | 916 | 916 |
| | 797 | 774 | 811 | 788 | 776 | 809 | 807 | 820 | 845 | 875 | 895 | 927 | 917 | 919 | 931 | 935 | 936 | 918 | 901 | 901 | 915 | 906 | 909 | 908 | 910 | 915 | 912 | 916 | 916 |
| | 738 | 738 | 706 | 704 | 704 | 717 | 720 | 759 | 752 | 753 | 770 | 770 | 770 | 758 | 760 | 790 | 783 | 795 | 802 | 810 | 812 | 809 | 852 | 862 | 895 | 905 | 921 | 917 | 916 |

$R_9 = 642$

$R_1 = 638$

$R_5 = 737$

$N_0 = 120$

TABLE 3 (Cont.)
Total Squared Errors of Prediction and Weight-Validities for Four Methods and a Single Criterion

| Methods | First New Sample | | | | | Second New Sample | | | | | Total Errors | | | | |
|---------|-------------------|-----|-----|-----|--|-------------------|-----|-----|-----|--|--------------|------|-----|------|--|
| | Weight-Validities | | | | | Weight-Validities | | | | | Total Errors | | | | |
| | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | | 1 | 2 | 3 | 5 | |
| 1 | 253 | 335 | 513 | 513 | | 350 | 480 | 592 | 592 | | 885 | 770 | 655 | 655 | |
| 2 | 589 | 376 | 497 | 425 | | 474 | 522 | 591 | 532 | | 800 | 738 | 656 | 723 | |
| 3 | 475 | 341 | 485 | 434 | | 549 | 504 | 566 | 559 | | 727 | 808 | 680 | 706 | |
| 4 | 436 | 428 | 482 | 426 | | 540 | 555 | 586 | 565 | | 794 | 755 | 660 | 709 | |
| 5 | 419 | 445 | 484 | 425 | | 522 | 565 | 588 | 534 | | 846 | 757 | 657 | 821 | |
| 6 | 592 | 422 | 491 | 435 | | 489 | 541 | 591 | 545 | | 925 | 809 | 654 | 816 | |
| 7 | 595 | 409 | 456 | 434 | | 521 | 526 | 570 | 537 | | 865 | 840 | 684 | 849 | |
| 8 | 590 | 407 | 450 | 412 | | 537 | 542 | 568 | 525 | | 870 | 809 | 687 | 877 | |
| 9 | 592 | 383 | 451 | 405 | | 551 | 582 | 568 | 521 | | 841 | 835 | 686 | 883 | |
| 10 | 556 | 392 | 458 | 399 | | 516 | 531 | 569 | 518 | | 924 | 868 | 686 | 906 | |
| 11 | 363 | 380 | 441 | 393 | | 518 | 545 | 557 | 518 | | 918 | 873 | 703 | 902 | |
| 12 | 357 | 394 | 432 | 377 | | 518 | 537 | 566 | 486 | | 913 | 913 | 695 | 987 | |
| 13 | 374 | 391 | 426 | 372 | | 516 | 539 | 560 | 503 | | 931 | 899 | 716 | 949 | |
| 14 | 398 | 366 | 388 | 375 | | 528 | 523 | 532 | 504 | | 905 | 929 | 792 | 954 | |
| 15 | 411 | 378 | 388 | 370 | | 530 | 524 | 533 | 511 | | 902 | 920 | 791 | 955 | |
| 16 | 400 | 393 | 389 | 376 | | 524 | 531 | 520 | 501 | | 903 | 893 | 816 | 981 | |
| 17 | 391 | 391 | 400 | 477 | | 527 | 527 | 537 | 492 | | 901 | 901 | 799 | 1001 | |
| 18 | 393 | 393 | 400 | 382 | | 528 | 528 | 536 | 492 | | 914 | 914 | 801 | 1005 | |
| 19 | 398 | 392 | 404 | 384 | | 527 | 521 | 540 | 495 | | 917 | 932 | 797 | 1003 | |
| 20 | 395 | 392 | 410 | 389 | | 520 | 518 | 543 | 498 | | 935 | 939 | 791 | 997 | |
| 21 | 395 | 388 | 411 | 384 | | 515 | 507 | 543 | 493 | | 945 | 968 | 790 | 1014 | |
| 22 | 396 | 390 | 390 | 380 | | 511 | 510 | 540 | 492 | | 958 | 972 | 813 | 1016 | |
| 23 | 387 | 386 | 390 | 381 | | 499 | 505 | 544 | 491 | | 978 | 987 | 809 | 1016 | |
| 24 | 390 | 387 | 381 | 385 | | 504 | 506 | 516 | 493 | | 978 | 983 | 872 | 1016 | |
| 25 | 387 | 393 | 376 | 388 | | 493 | 508 | 512 | 493 | | 1014 | 980 | 883 | 1021 | |
| 26 | 386 | 397 | 381 | 385 | | 490 | 504 | 504 | 494 | | 1031 | 994 | 984 | 1020 | |
| 27 | 388 | 391 | 384 | 388 | | 490 | 494 | 504 | 493 | | 1031 | 1013 | 989 | 1023 | |
| 28 | 389 | 389 | 385 | 389 | | 491 | 491 | 505 | 493 | | 1029 | 1029 | 988 | 1022 | |
| 29 | 389 | | | | | 491 | | | | | 1030 | | | | |
| | $N_0 = 75$ | | | | | $R_1 = 608$ | | | | | $R_2 = 684$ | | | | |
| | $R_0 = 854$ | | | | | | | | | | | | | | |

Ranks

the outcome of the latter would presumably be much more subject to random variability of weight-validities from rank to rank.

In Table 3 are presented data from ten additional original samples from the criterion-1 (All-University) population, with sizes ranging from 435 down to 30 cases. Here all sets of weights from each original sample were cross-validated on two new samples, where again each new sample consisted of 252 cases. Total squared errors of prediction are presented as well as weight-validities for each of the 20 new samples. Method 4 was omitted from this phase of the computations. At the bottom of each page of Table 3 are given, in addition to the original sample size N_0 , the full-rank multiple correlations for the three samples represented by that page; these are denoted by R_0 , R_1 and R_2 for the original sample, first new sample, and second new sample, respectively.

Since the criterion variable (as well as the predictors) was normalized before the computations were carried out, the total squared errors of prediction are comparable from sample to sample as well as from method to method and rank to rank. Expressed in normal deviates, the criterion mean is zero and the sum of squares is one. Thus if a prediction of zero were made for each case, without ever going to the trouble of computing regression weights, the total squared errors of prediction would be one. Since, for example, the total squared errors of prediction using the full-rank weights from an original sample of size 75 are greater than one in both new samples, it appears that this particular regression equation is worse than useless. Yet for this same sample the rank-1 errors for method 3 of .737 and .655 are actually lower than either of the full-rank errors obtained for the sample of 390 cases, which were .767 and .745. In general, it may be seen that the lower-rank errors obtained with method 3 using small original samples compare favorably, or at least not unfavorably, with the full-rank errors obtained using large original samples. A similar trend may be noted, though not so clearly, with regard to weight-validities.

Table 4 was prepared from Table 3 in a manner analogous to the preparation of Table 2 from Table 1. Here, of course, only one criterion variable is involved, and the comparisons are made with respect to total squared errors of prediction as well as to weight-validities. For the larger original-sample sizes, the outcomes of the comparisons are not appreciably affected by the index of accuracy used. For the smaller sizes, however, the total squared errors of prediction tend to favor method 3 over the other methods and the lower ranks over the higher to a greater extent than do the weight-validities. In the present series of samples, just as in the preceding series, method 3 appears to be definitely superior to the other methods. And even for the largest original-sample sizes, method 3 appears preferable to the full-rank system.

It appears that method 3 could be used to considerable advantage in

TABLE 4
Comparison Between Four Reduced-Rank Methods With Respect to Weight-Validities
and Total Squared Errors of Prediction for a Single Criterion

| Sample Size | Methods | Index | Number of ranks for which index is superior to other methods | | | | Number of ranks for which index is superior to full-rank method | | | |
|----------------|---------|-------|--|----------|-------|----------|---|----------|-------|----------|
| | | | W_1 | ψ_1 | W_2 | ψ_2 | W_1 | ψ_1 | W_2 | ψ_2 |
| 435 | 1 | | 2.33 | 2.33 | .25 | 0. | 6.5 | 6.5 | 10.5 | 11 |
| | 2 | | 3.33 | 3.83 | .25 | 0. | 3.5 | 5. | 8.5 | 10 |
| | 3 | | 21.5 | 21. | 26.75 | 27.5 | 18. | 20. | 27.5 | 28 |
| | 5 | | .83 | .83 | .75 | .5 | 0. | 0. | 17.5 | 19.5 |
| 390 | 1 | | 1.33 | 1. | 0. | 0. | 8. | 7.5 | 2. | 2. |
| | 2 | | 2.33 | 2. | .33 | .5 | 18.5 | 18.5 | 6.5 | 8. |
| | 3 | | 19. | 20.5 | 14.33 | 15.5 | 24. | 24.5 | 20.5 | 22.5 |
| | 5 | | 5.33 | 4.5 | 13.33 | 12. | 19. | 17. | 24. | 25. |
| 345 | 1 | | .83 | .5 | 2.5 | 1.5 | 12. | 10. | 20.5 | 24. |
| | 2 | | 3.83 | 3.5 | 3.5 | 4. | 10. | 9.5 | 20. | 22. |
| | 3 | | 20.83 | 21.5 | 20.5 | 22. | 18. | 21. | 27. | 27. |
| | 5 | | 2.5 | 2.5 | 1.5 | .5 | 5. | 4. | 20.5 | 21.5 |
| 300 | 1 | | 1. | 1. | 2. | 2. | 6.5 | 5. | 6. | 6. |
| | 2 | | 0. | 0. | 3. | 3. | 12.5 | 12. | 11.5 | 11.5 |
| | 3 | | 24. | 24. | 18. | 18. | 27. | 27. | 20. | 20. |
| | 5 | | 3. | 3. | 5. | 5. | 20.5 | 20.5 | 16. | 16.5 |
| 255 | 1 | | 1. | 1. | 2.5 | 2. | 11.5 | 13.5 | 16.5 | 14. |
| | 2 | | 2. | 2. | 10.5 | 8. | 13. | 14.5 | 19.5 | 20.5 |
| | 3 | | 23. | 23. | 4. | 6.5 | 24. | 24. | 8. | 21. |
| | 5 | | 2. | 2. | 11. | 11.5 | 14.5 | 14.5 | 21.5 | 27. |
| 210 | 1 | | .33 | .33 | 2. | 1.5 | 3. | 5.5 | 8. | 13.5 |
| | 2 | | .33 | 1.33 | 2. | 1.5 | 5.5 | 9.5 | 9.5 | 14.5 |
| | 3 | | 21. | 22. | 21.5 | 22.5 | 24. | 24. | 25. | 26. |
| | 5 | | 6.33 | 4.33 | 2.5 | 2.5 | 18.5 | 21.5 | 14.5 | 14 |
| 165 | 1 | | 4.33 | 4.5 | 0. | 0. | 7. | 8.5 | 18.5 | 20.5 |
| | 2 | | 3.83 | 5.5 | 1. | 1. | 4. | 6.5 | 19.5 | 24. |
| | 3 | | 11.5 | 14.5 | 26.5 | 26.5 | 15. | 21. | 27. | 27. |
| | 5 | | 8.33 | 3.5 | .5 | .5 | 22.5 | 22. | 6.5 | 8. |
| 120 | 1 | | 1. | 1. | 1.5 | 0. | 15. | 19.5 | 19.5 | 20. |
| | 2 | | 0. | 0. | 2.5 | 2. | 11. | 13. | 14.5 | 16. |
| | 3 | | 26.5 | 26.5 | 22.5 | 24.5 | 27. | 27. | 24. | 26. |
| | 5 | | .5 | .5 | 1.5 | 1.5 | 16. | 17.5 | 23.5 | 25.5 |
| 75 | 1 | | 5.33 | 0. | 0. | 0. | 18. | 27.5 | 26. | 23.5 |
| | 2 | | 3.33 | 1. | 1.5 | 0. | 17.5 | 27. | 28. | 26.5 |
| | 3 | | 18.5 | 26.5 | 26. | 27.5 | 20.5 | 27. | 28. | 28. |
| | 5 | | .83 | .5 | .5 | .5 | 12. | 25. | 28. | 26.5 |
| 30 | 1 | | 0. | 0. | 0. | 0. | 27. | 28. | 27. | 28. |
| | 2 | | 9. | 0. | 2. | 0. | 28. | 28. | 28. | 28. |
| | 3 | | 17.5 | 26.5 | 25. | 26.5 | 28. | 28. | 28. | 28. |
| | 5 | | 1.5 | 1.5 | 1. | 1.5 | 25.5 | 25. | 24. | 26. |

either of two situations. The first would be where, for a given original-sample size, one wanted the greatest accuracy of prediction obtainable. The other would be where, for a given accuracy of prediction, one wanted to use the smallest possible original sample. In order actually to compute the coefficients for a reduced-rank prediction equation, however, one has, of course, to select the particular rank to be used. To provide some indication as to how satisfactory the statistics \hat{W} and $\hat{\psi}$ would be for this purpose, they are computed for the original samples of Table 3 using (46) and (96), respectively. They were computed only for method 3, since the other methods are dependent on the criterion observations for order of selection, contrary to the assumptions used in deriving the above statistics. These estimated values for weight-validities and total squared errors of prediction are given in Table 5. To facilitate comparisons, the obtained values from Table 3 are reproduced in the adjacent columns. At the bottom of each page are given the original-sample size and the full-rank multiple correlations for the two cross-validation samples. The multiple correlation and the estimated population correlation, from (32), in the original sample are given for each rank. The column headed $\hat{\sigma}$ is an estimate of the standard error of $\hat{\psi}$, and may be derived as follows. We let a be a column vector composed of the elements of z_2 and z_3 in (91). Then we may write

$$(135) \quad \hat{\psi} = \frac{N+L}{N-L} a'a,$$

where the elements a_i of a are independently distributed with mean zero and variance σ^2 . The variance of $a'a$ will be

$$(136) \quad \text{Var}(a'a) = E[(a'a)^2] - [E(a'a)]^2.$$

Under the reduced-rank hypothesis, $a'a$ will be simply the error sum of squares in the original sample, so that from (71), the second term on the right of (136) will be

$$(137) \quad [E(a'a)]^2 = [(N-L)\sigma^2]^2 = (N-L)^2\sigma^4.$$

Expanding the first term on the right of (136), we obtain

$$(138) \quad E[(a'a)^2] = (N-L)E(a_i^4) + (N-L)(N-L-1)E(a_i^2a_j^2), \quad i \neq j.$$

Since the a_i are independent, we have

$$(139) \quad E(a_i^2a_j^2) = E(a_i^2)E(a_j^2) = \sigma^4, \quad i \neq j.$$

If the elements of the criterion vector, y , are assumed to be normally distributed, the elements of a , being linear combinations of the criterion observations, will also be normally distributed. Thus we have (Cramér, 1946, p. 212):

$$(140) \quad E(a_i^4) = 3\sigma^4.$$

TABLE 5
Estimated and Obtained Measures of Accuracy of Prediction Using
Method of Largest Principal-Axes Factors

| | R_0 | R_c | α | \hat{W} | W_1 | W_2 | $\hat{\psi}$ | ψ_1 | ψ_2 |
|----|-------------|-------|----------|-------------|-------|-------|--------------|----------|----------|
| 1 | 539 | 538 | 048 | 536 | 582 | 488 | 712 | 663 | 763 |
| 2 | 549 | 546 | 048 | 543 | 596 | 487 | 705 | 647 | 764 |
| 3 | 549 | 545 | 048 | 540 | 596 | 487 | 708 | 647 | 765 |
| 4 | 550 | 544 | 048 | 538 | 599 | 491 | 711 | 643 | 760 |
| 5 | 558 | 551 | 048 | 543 | 603 | 499 | 705 | 638 | 753 |
| 6 | 559 | 550 | 048 | 542 | 608 | 503 | 707 | 633 | 749 |
| 7 | 568 | 558 | 048 | 548 | 619 | 513 | 700 | 619 | 738 |
| 8 | 568 | 556 | 048 | 545 | 620 | 513 | 703 | 617 | 738 |
| 9 | 568 | 555 | 048 | 543 | 620 | 515 | 706 | 617 | 737 |
| 10 | 568 | 554 | 049 | 540 | 620 | 516 | 709 | 618 | 736 |
| 11 | 571 | 555 | 049 | 540 | 613 | 514 | 709 | 625 | 738 |
| 12 | 571 | 554 | 049 | 537 | 613 | 514 | 712 | 625 | 738 |
| 13 | 571 | 552 | 049 | 534 | 613 | 514 | 716 | 625 | 738 |
| 14 | 571 | 551 | 049 | 532 | 614 | 511 | 718 | 624 | 741 |
| 15 | 578 | 557 | 049 | 536 | 613 | 507 | 714 | 625 | 747 |
| 16 | 583 | 561 | 049 | 539 | 617 | 518 | 711 | 620 | 734 |
| 17 | 584 | 561 | 049 | 538 | 617 | 514 | 713 | 619 | 739 |
| 18 | 590 | 566 | 049 | 543 | 624 | 525 | 708 | 611 | 729 |
| 19 | 593 | 568 | 049 | 543 | 622 | 526 | 707 | 613 | 728 |
| 20 | 594 | 567 | 049 | 542 | 629 | 521 | 709 | 604 | 734 |
| 21 | 609 | 582 | 048 | 556 | 614 | 491 | 693 | 623 | 776 |
| 22 | 611 | 583 | 048 | 557 | 619 | 500 | 693 | 617 | 767 |
| 23 | 615 | 586 | 048 | 558 | 617 | 496 | 691 | 620 | 774 |
| 24 | 617 | 587 | 048 | 558 | 621 | 492 | 692 | 616 | 780 |
| 25 | 619 | 588 | 048 | 558 | 615 | 488 | 692 | 623 | 787 |
| 26 | 619 | 587 | 048 | 556 | 615 | 486 | 695 | 624 | 789 |
| 27 | 622 | 589 | 048 | 557 | 610 | 478 | 694 | 630 | 797 |
| 28 | 625 | 590 | 048 | 558 | 608 | 472 | 693 | 633 | 805 |
| 29 | 626 | 591 | 049 | 557 | 613 | 472 | 694 | 626 | 806 |
| | $N_0 = 435$ | | | $R_1 = 684$ | | | $R_2 = 582$ | | |

Decimal point preceding each entry has been omitted.

TABLE 5 (Cont.)
 Estimated and Obtained Measures of Accuracy of Prediction Using
 Method of Largest Principal-Axes Factors

| | R_0 | R_c | α | \hat{W} | W_1 | W_2 | $\hat{\psi}$ | ψ_1 | ψ_2 | |
|-------|-------------|-------|----------|-------------|-------|-------|--------------|----------|----------|-----|
| Ranks | 1 | 545 | 544 | 051 | 542 | 481 | 502 | 706 | 770 | 749 |
| | 2 | 554 | 550 | 050 | 547 | 507 | 516 | 701 | 743 | 734 |
| | 3 | 554 | 549 | 050 | 544 | 506 | 514 | 704 | 744 | 736 |
| | 4 | 562 | 555 | 050 | 549 | 518 | 535 | 699 | 732 | 714 |
| | 5 | 562 | 554 | 050 | 546 | 519 | 535 | 702 | 731 | 714 |
| | 6 | 568 | 559 | 050 | 550 | 530 | 548 | 698 | 720 | 700 |
| | 7 | 571 | 560 | 050 | 550 | 519 | 533 | 698 | 731 | 716 |
| | 8 | 578 | 565 | 050 | 553 | 518 | 535 | 694 | 733 | 715 |
| | 9 | 585 | 571 | 050 | 558 | 518 | 516 | 689 | 733 | 737 |
| | 10 | 586 | 571 | 050 | 556 | 516 | 514 | 692 | 737 | 740 |
| | 11 | 587 | 571 | 050 | 555 | 514 | 508 | 693 | 740 | 747 |
| | 12 | 587 | 569 | 051 | 552 | 517 | 510 | 697 | 736 | 746 |
| | 13 | 588 | 568 | 051 | 549 | 518 | 510 | 700 | 736 | 746 |
| | 14 | 588 | 567 | 051 | 546 | 518 | 509 | 703 | 736 | 746 |
| | 15 | 591 | 569 | 051 | 547 | 522 | 518 | 703 | 731 | 738 |
| | 16 | 592 | 568 | 051 | 545 | 516 | 523 | 705 | 738 | 732 |
| | 17 | 595 | 570 | 051 | 546 | 528 | 536 | 704 | 724 | 717 |
| | 18 | 605 | 579 | 051 | 554 | 522 | 542 | 695 | 730 | 713 |
| | 19 | 607 | 579 | 051 | 553 | 525 | 540 | 697 | 728 | 715 |
| | 20 | 610 | 582 | 051 | 554 | 502 | 535 | 696 | 755 | 722 |
| | 21 | 611 | 581 | 052 | 552 | 495 | 533 | 699 | 763 | 723 |
| | 22 | 611 | 579 | 052 | 550 | 497 | 532 | 702 | 761 | 725 |
| | 23 | 614 | 582 | 052 | 551 | 496 | 530 | 701 | 762 | 727 |
| | 24 | 615 | 581 | 052 | 549 | 497 | 525 | 703 | 761 | 733 |
| | 25 | 619 | 583 | 052 | 550 | 496 | 515 | 702 | 766 | 745 |
| | 26 | 619 | 582 | 052 | 548 | 495 | 516 | 705 | 767 | 743 |
| | 27 | 619 | 581 | 052 | 545 | 492 | 517 | 709 | 771 | 743 |
| | 28 | 619 | 579 | 053 | 542 | 491 | 516 | 712 | 772 | 743 |
| | 29 | 619 | 578 | 053 | 539 | 495 | 515 | 715 | 767 | 745 |
| | $N_0 = 390$ | | | $R_1 = 646$ | | | $R_2 = 638$ | | | |

TABLE 5 (Cont.)
 Estimated and Obtained Measures of Accuracy of Prediction Using
 Method of Largest Principal-Axes Factors

| | R_0 | R_c | α | \hat{W} | W_1 | W_2 | $\hat{\psi}$ | ψ_1 | ψ_2 | |
|-------------|-------|-------|----------|-------------|-------|-------|--------------|-------------|----------|-----|
| Ranks | 1 | 598 | 596 | 049 | 595 | 511 | 531 | 646 | 747 | 723 |
| | 2 | 601 | 598 | 049 | 595 | 524 | 535 | 646 | 732 | 718 |
| | 3 | 605 | 600 | 049 | 596 | 518 | 530 | 645 | 741 | 724 |
| | 4 | 622 | 616 | 048 | 610 | 516 | 524 | 628 | 753 | 735 |
| | 5 | 625 | 618 | 048 | 611 | 523 | 530 | 627 | 742 | 727 |
| | 6 | 627 | 618 | 048 | 610 | 527 | 536 | 629 | 735 | 720 |
| | 7 | 628 | 618 | 048 | 608 | 530 | 536 | 630 | 731 | 721 |
| | 8 | 630 | 618 | 049 | 607 | 534 | 535 | 632 | 726 | 722 |
| | 9 | 630 | 617 | 049 | 604 | 535 | 535 | 636 | 726 | 722 |
| | 10 | 633 | 619 | 049 | 605 | 537 | 532 | 635 | 724 | 727 |
| | 11 | 634 | 619 | 049 | 603 | 536 | 531 | 637 | 727 | 730 |
| | 12 | 641 | 624 | 049 | 608 | 534 | 513 | 631 | 735 | 756 |
| | 13 | 643 | 625 | 049 | 607 | 531 | 506 | 633 | 738 | 768 |
| | 14 | 643 | 623 | 049 | 604 | 530 | 505 | 636 | 739 | 768 |
| | 15 | 652 | 631 | 049 | 612 | 549 | 516 | 628 | 716 | 759 |
| | 16 | 654 | 633 | 049 | 612 | 546 | 507 | 627 | 722 | 771 |
| | 17 | 658 | 635 | 049 | 613 | 543 | 510 | 626 | 726 | 766 |
| | 18 | 659 | 635 | 049 | 611 | 541 | 513 | 628 | 728 | 762 |
| | 19 | 659 | 633 | 049 | 609 | 541 | 512 | 632 | 728 | 763 |
| | 20 | 661 | 634 | 049 | 609 | 553 | 519 | 632 | 712 | 752 |
| | 21 | 664 | 636 | 049 | 609 | 547 | 521 | 632 | 719 | 750 |
| | 22 | 666 | 637 | 050 | 610 | 550 | 518 | 632 | 716 | 756 |
| | 23 | 666 | 636 | 050 | 607 | 552 | 519 | 635 | 712 | 755 |
| | 24 | 668 | 637 | 050 | 607 | 548 | 518 | 636 | 717 | 757 |
| | 25 | 673 | 641 | 050 | 610 | 535 | 509 | 632 | 732 | 769 |
| | 26 | 673 | 639 | 050 | 607 | 535 | 509 | 636 | 732 | 769 |
| | 27 | 674 | 639 | 050 | 606 | 535 | 503 | 638 | 732 | 778 |
| | 28 | 675 | 639 | 051 | 604 | 532 | 497 | 640 | 737 | 788 |
| | 29 | 676 | 638 | 051 | 602 | 533 | 502 | 642 | 736 | 782 |
| $N_0 = 345$ | | | | $R_1 = 649$ | | | | $R_2 = 630$ | | |

TABLE 5 (Cont.)
 Estimated and Obtained Measures of Accuracy of Prediction Using
 Method of Largest Principal-Axes Factors

| | | R_0 | R_c | $\hat{\alpha}$ | \hat{W} | W_1 | W_2 | $\hat{\psi}$ | ψ_1 | ψ_2 |
|-------------|----|-------|-------|----------------|-----------|-------|-------|--------------|----------|----------|
| Ranks | 1 | 493 | 490 | 062 | 487 | 561 | 542 | 762 | 691 | 710 |
| | 2 | 524 | 519 | 060 | 515 | 556 | 545 | 735 | 692 | 704 |
| | 3 | 524 | 517 | 060 | 511 | 557 | 548 | 740 | 691 | 701 |
| | 4 | 525 | 516 | 061 | 506 | 558 | 546 | 744 | 689 | 702 |
| | 5 | 552 | 541 | 059 | 531 | 564 | 540 | 719 | 682 | 708 |
| | 6 | 553 | 540 | 059 | 528 | 560 | 539 | 722 | 686 | 710 |
| | 7 | 553 | 538 | 060 | 523 | 562 | 540 | 727 | 685 | 709 |
| | 8 | 559 | 542 | 060 | 525 | 560 | 554 | 725 | 686 | 694 |
| | 9 | 559 | 540 | 060 | 521 | 560 | 554 | 730 | 686 | 694 |
| | 10 | 563 | 542 | 060 | 521 | 547 | 565 | 730 | 701 | 682 |
| | 11 | 564 | 540 | 061 | 518 | 551 | 566 | 734 | 697 | 681 |
| | 12 | 566 | 540 | 061 | 515 | 548 | 569 | 737 | 700 | 677 |
| | 13 | 568 | 540 | 061 | 513 | 547 | 566 | 739 | 701 | 681 |
| | 14 | 568 | 538 | 062 | 510 | 551 | 563 | 743 | 696 | 683 |
| | 15 | 577 | 545 | 062 | 516 | 544 | 571 | 738 | 704 | 675 |
| | 16 | 579 | 546 | 062 | 515 | 546 | 574 | 739 | 702 | 671 |
| | 17 | 583 | 548 | 062 | 515 | 547 | 580 | 739 | 701 | 665 |
| | 18 | 590 | 554 | 062 | 520 | 568 | 585 | 735 | 677 | 659 |
| | 19 | 593 | 554 | 062 | 519 | 560 | 578 | 736 | 686 | 666 |
| | 20 | 593 | 553 | 062 | 515 | 557 | 575 | 741 | 690 | 670 |
| | 21 | 595 | 553 | 063 | 513 | 554 | 571 | 743 | 694 | 674 |
| | 22 | 595 | 550 | 063 | 509 | 553 | 573 | 748 | 694 | 672 |
| | 23 | 596 | 549 | 064 | 506 | 553 | 570 | 752 | 694 | 675 |
| | 24 | 598 | 549 | 064 | 504 | 548 | 572 | 755 | 701 | 673 |
| | 25 | 598 | 546 | 065 | 500 | 547 | 572 | 760 | 702 | 673 |
| | 26 | 604 | 552 | 064 | 504 | 525 | 559 | 755 | 726 | 688 |
| | 27 | 606 | 552 | 065 | 503 | 529 | 551 | 758 | 721 | 697 |
| | 28 | 607 | 550 | 065 | 499 | 529 | 546 | 762 | 722 | 703 |
| | 29 | 608 | 549 | 066 | 496 | 527 | 550 | 766 | 723 | 698 |
| $N_0 = 300$ | | | | $R_1 = 664$ | | | | $R_2 = 664$ | | |

TABLE 5 (Cont.)
Estimated and Obtained Measures of Accuracy of Prediction Using
Method of Largest Principal-Axes Factors

| | R_0 | R_c | α | \hat{W} | W_1 | W_2 | $\hat{\psi}$ | ψ_1 | ψ_2 | |
|-------------|-------|-------|----------|-------------|-------|-------|--------------|-------------|----------|-----|
| Ranks | 1 | 559 | 557 | 061 | 555 | 561 | 473 | 692 | 687 | 780 |
| | 2 | 593 | 588 | 058 | 584 | 592 | 443 | 659 | 650 | 820 |
| | 3 | 593 | 587 | 059 | 580 | 590 | 443 | 663 | 652 | 820 |
| | 4 | 593 | 585 | 059 | 576 | 590 | 443 | 669 | 652 | 819 |
| | 5 | 595 | 584 | 060 | 573 | 597 | 446 | 672 | 644 | 817 |
| | 6 | 596 | 583 | 060 | 570 | 596 | 439 | 676 | 645 | 825 |
| | 7 | 599 | 583 | 061 | 568 | 612 | 446 | 678 | 627 | 817 |
| | 8 | 599 | 581 | 061 | 564 | 611 | 446 | 683 | 627 | 818 |
| | 9 | 601 | 581 | 062 | 562 | 612 | 446 | 686 | 626 | 821 |
| | 10 | 601 | 579 | 062 | 557 | 612 | 447 | 691 | 626 | 821 |
| | 11 | 601 | 577 | 063 | 553 | 614 | 449 | 696 | 624 | 818 |
| | 12 | 602 | 576 | 063 | 550 | 608 | 441 | 700 | 631 | 827 |
| | 13 | 604 | 575 | 064 | 547 | 607 | 436 | 704 | 633 | 832 |
| | 14 | 605 | 574 | 064 | 545 | 616 | 435 | 707 | 622 | 834 |
| | 15 | 607 | 573 | 065 | 542 | 612 | 438 | 711 | 625 | 832 |
| | 16 | 608 | 572 | 065 | 539 | 608 | 441 | 714 | 630 | 830 |
| | 17 | 612 | 575 | 065 | 540 | 608 | 435 | 714 | 631 | 836 |
| | 18 | 618 | 579 | 065 | 542 | 620 | 442 | 711 | 616 | 828 |
| | 19 | 623 | 582 | 065 | 544 | 602 | 431 | 710 | 638 | 841 |
| | 20 | 623 | 580 | 066 | 540 | 600 | 435 | 716 | 640 | 840 |
| | 21 | 626 | 581 | 066 | 539 | 599 | 435 | 717 | 642 | 839 |
| | 22 | 639 | 593 | 065 | 551 | 600 | 443 | 704 | 640 | 833 |
| | 23 | 640 | 593 | 065 | 549 | 589 | 449 | 707 | 654 | 828 |
| | 24 | 641 | 591 | 066 | 545 | 591 | 449 | 712 | 652 | 829 |
| | 25 | 644 | 592 | 066 | 545 | 581 | 452 | 713 | 667 | 828 |
| | 26 | 645 | 592 | 067 | 543 | 573 | 457 | 716 | 676 | 821 |
| | 27 | 645 | 590 | 067 | 538 | 572 | 457 | 722 | 678 | 822 |
| | 28 | 646 | 587 | 068 | 534 | 569 | 456 | 727 | 682 | 823 |
| | 29 | 646 | 586 | 069 | 531 | 576 | 448 | 732 | 673 | 832 |
| $N_0 = 225$ | | | | $R_1 = 717$ | | | | $R_2 = 563$ | | |

TABLE 5 (Cont.)
 Estimated and Obtained Measures of Accuracy of Prediction Using
 Method of Largest Principal-Axes Factors

| | R_0 | R_c | α | \hat{W} | W_1 | W_2 | $\hat{\psi}$ | ψ_1 | ψ_2 | |
|-------------|-------|-------|----------|-------------|-------|-------|--------------|----------|----------|-----|
| Ranks | 1 | 528 | 525 | 071 | 522 | 498 | 571 | 728 | 753 | 676 |
| | 2 | 537 | 530 | 071 | 524 | 506 | 566 | 726 | 745 | 680 |
| | 3 | 538 | 528 | 072 | 519 | 507 | 563 | 731 | 744 | 684 |
| | 4 | 538 | 525 | 072 | 512 | 508 | 563 | 738 | 743 | 683 |
| | 5 | 546 | 531 | 072 | 515 | 498 | 565 | 736 | 754 | 681 |
| | 6 | 583 | 566 | 069 | 550 | 525 | 572 | 699 | 727 | 673 |
| | 7 | 601 | 582 | 067 | 564 | 519 | 570 | 683 | 740 | 677 |
| | 8 | 607 | 586 | 067 | 566 | 522 | 576 | 682 | 736 | 670 |
| | 9 | 607 | 583 | 068 | 561 | 521 | 575 | 688 | 737 | 671 |
| | 10 | 608 | 581 | 069 | 556 | 522 | 573 | 694 | 735 | 673 |
| | 11 | 609 | 580 | 070 | 552 | 521 | 571 | 698 | 737 | 677 |
| | 12 | 611 | 579 | 070 | 549 | 526 | 569 | 702 | 732 | 679 |
| | 13 | 616 | 581 | 070 | 549 | 520 | 552 | 703 | 740 | 701 |
| | 14 | 616 | 579 | 071 | 544 | 519 | 554 | 710 | 743 | 699 |
| | 15 | 632 | 594 | 070 | 558 | 509 | 560 | 694 | 762 | 695 |
| | 16 | 633 | 593 | 071 | 555 | 514 | 561 | 698 | 756 | 695 |
| | 17 | 639 | 596 | 071 | 557 | 500 | 554 | 696 | 774 | 705 |
| | 18 | 647 | 603 | 070 | 562 | 499 | 539 | 691 | 773 | 725 |
| | 19 | 647 | 601 | 071 | 558 | 499 | 538 | 697 | 774 | 727 |
| | 20 | 647 | 598 | 072 | 553 | 499 | 540 | 704 | 773 | 725 |
| | 21 | 651 | 600 | 072 | 553 | 504 | 532 | 704 | 766 | 732 |
| | 22 | 653 | 599 | 073 | 550 | 507 | 542 | 708 | 762 | 720 |
| | 23 | 653 | 597 | 074 | 545 | 506 | 542 | 715 | 764 | 722 |
| | 24 | 658 | 599 | 074 | 546 | 500 | 542 | 714 | 775 | 724 |
| | 25 | 660 | 600 | 074 | 545 | 488 | 539 | 716 | 795 | 730 |
| | 26 | 664 | 602 | 074 | 546 | 490 | 523 | 716 | 794 | 754 |
| | 27 | 665 | 600 | 075 | 541 | 486 | 519 | 722 | 798 | 759 |
| | 28 | 665 | 597 | 076 | 536 | 486 | 520 | 729 | 797 | 758 |
| | 29 | 670 | 600 | 076 | 538 | 493 | 524 | 728 | 789 | 757 |
| $N_0 = 210$ | | | | $R_1 = 605$ | | | $R_2 = 653$ | | | |

TABLE 5 (Cont.)
 Estimated and Obtained Measures of Accuracy of Prediction Using
 Method of Largest Principal-Axes Factors

| | R_0 | R_c | $\hat{\alpha}$ | \hat{W} | W_1 | W_2 | $\hat{\psi}$ | ψ_1 | ψ_2 | |
|-------------|-------|-------|----------------|-------------|-------|-------|--------------|----------|----------|-----|
| Ranks | 1 | 544 | 540 | 078 | 536 | 587 | 540 | 712 | 658 | 709 |
| | 2 | 544 | 536 | 079 | 528 | 588 | 541 | 721 | 657 | 708 |
| | 3 | 549 | 537 | 080 | 525 | 581 | 543 | 725 | 665 | 705 |
| | 4 | 563 | 548 | 079 | 533 | 591 | 537 | 716 | 652 | 712 |
| | 5 | 582 | 564 | 078 | 546 | 579 | 539 | 703 | 665 | 710 |
| | 6 | 590 | 569 | 078 | 548 | 578 | 534 | 701 | 666 | 716 |
| | 7 | 593 | 569 | 079 | 545 | 582 | 547 | 705 | 661 | 702 |
| | 8 | 608 | 581 | 078 | 555 | 584 | 537 | 695 | 660 | 716 |
| | 9 | 618 | 588 | 078 | 560 | 573 | 538 | 690 | 676 | 717 |
| | 10 | 618 | 585 | 079 | 553 | 574 | 541 | 698 | 675 | 714 |
| | 11 | 639 | 605 | 077 | 573 | 563 | 527 | 677 | 696 | 736 |
| | 12 | 645 | 609 | 077 | 574 | 557 | 522 | 675 | 707 | 743 |
| | 13 | 645 | 606 | 078 | 568 | 559 | 523 | 683 | 704 | 741 |
| | 14 | 646 | 603 | 079 | 562 | 555 | 522 | 691 | 710 | 744 |
| | 15 | 648 | 601 | 080 | 558 | 555 | 529 | 697 | 712 | 735 |
| | 16 | 648 | 598 | 081 | 552 | 554 | 529 | 705 | 713 | 735 |
| | 17 | 648 | 594 | 082 | 545 | 550 | 528 | 713 | 719 | 736 |
| | 18 | 649 | 591 | 084 | 539 | 550 | 529 | 721 | 719 | 735 |
| | 19 | 649 | 588 | 085 | 533 | 552 | 534 | 729 | 717 | 729 |
| | 20 | 650 | 585 | 086 | 527 | 558 | 538 | 737 | 708 | 722 |
| | 21 | 651 | 583 | 087 | 522 | 550 | 536 | 744 | 721 | 725 |
| | 22 | 657 | 586 | 087 | 523 | 543 | 532 | 744 | 733 | 731 |
| | 23 | 657 | 582 | 089 | 516 | 543 | 532 | 753 | 733 | 731 |
| | 24 | 658 | 580 | 090 | 511 | 544 | 527 | 761 | 732 | 741 |
| | 25 | 659 | 578 | 091 | 506 | 539 | 528 | 767 | 740 | 740 |
| | 26 | 659 | 573 | 093 | 499 | 539 | 528 | 777 | 740 | 740 |
| | 27 | 659 | 569 | 094 | 492 | 538 | 529 | 787 | 741 | 738 |
| | 28 | 665 | 573 | 094 | 494 | 551 | 499 | 786 | 727 | 778 |
| | 29 | 666 | 570 | 096 | 487 | 555 | 502 | 794 | 723 | 777 |
| $N_0 = 165$ | | | | $R_1 = 679$ | | | $R_2 = 646$ | | | |

TABLE 5 (Cont.)
 Estimated and Obtained Measures of Accuracy of Prediction Using
 Method of Largest Principal-Axes Factors

| | R_0 | R_e | $\hat{\alpha}$ | \hat{W} | W_1 | W_2 | $\hat{\psi}$ | ψ_1 | ψ_2 | |
|-------------|-------|-------|----------------|-------------|-------|-------|--------------|-------------|----------|-----|
| Ranks | 1 | 554 | 549 | 091 | 543 | 546 | 514 | 705 | 703 | 738 |
| | 2 | 582 | 572 | 088 | 563 | 536 | 543 | 684 | 718 | 706 |
| | 3 | 582 | 568 | 090 | 553 | 536 | 545 | 695 | 718 | 704 |
| | 4 | 582 | 563 | 092 | 543 | 534 | 545 | 706 | 720 | 704 |
| | 5 | 582 | 557 | 094 | 533 | 534 | 545 | 718 | 721 | 704 |
| | 6 | 597 | 568 | 093 | 540 | 541 | 535 | 711 | 713 | 717 |
| | 7 | 598 | 563 | 095 | 531 | 542 | 533 | 723 | 712 | 720 |
| | 8 | 607 | 569 | 096 | 533 | 521 | 501 | 721 | 743 | 759 |
| | 9 | 637 | 598 | 092 | 561 | 512 | 505 | 691 | 754 | 752 |
| | 10 | 638 | 594 | 094 | 553 | 516 | 504 | 701 | 750 | 753 |
| | 11 | 647 | 600 | 091 | 556 | 509 | 490 | 699 | 759 | 770 |
| | 12 | 647 | 595 | 096 | 547 | 509 | 490 | 711 | 759 | 770 |
| | 13 | 647 | 590 | 098 | 537 | 509 | 490 | 723 | 759 | 770 |
| | 14 | 660 | 601 | 097 | 547 | 520 | 503 | 713 | 752 | 758 |
| | 15 | 660 | 596 | 099 | 538 | 522 | 502 | 725 | 749 | 760 |
| | 16 | 674 | 609 | 098 | 549 | 506 | 480 | 714 | 775 | 790 |
| | 17 | 678 | 608 | 099 | 546 | 496 | 487 | 720 | 789 | 783 |
| | 18 | 683 | 610 | 100 | 544 | 479 | 479 | 723 | 816 | 795 |
| | 19 | 683 | 605 | 102 | 536 | 479 | 473 | 734 | 816 | 802 |
| | 20 | 699 | 622 | 100 | 553 | 484 | 474 | 716 | 817 | 810 |
| | 21 | 699 | 617 | 102 | 544 | 485 | 473 | 728 | 816 | 812 |
| | 22 | 703 | 617 | 104 | 541 | 475 | 480 | 733 | 835 | 809 |
| | 23 | 712 | 624 | 103 | 547 | 478 | 454 | 728 | 838 | 852 |
| | 24 | 713 | 622 | 105 | 541 | 482 | 450 | 736 | 833 | 862 |
| | 25 | 725 | 633 | 104 | 553 | 466 | 432 | 724 | 863 | 895 |
| | 26 | 726 | 630 | 106 | 546 | 462 | 428 | 734 | 870 | 905 |
| | 27 | 735 | 638 | 105 | 554 | 448 | 434 | 726 | 904 | 921 |
| | 28 | 737 | 635 | 107 | 548 | 441 | 440 | 735 | 917 | 917 |
| | 29 | 737 | 630 | 110 | 539 | 443 | 411 | 748 | 914 | 916 |
| $N_0 = 120$ | | | | $R_1 = 638$ | | | | $R_2 = 642$ | | |

TABLE 5 (Cont.)
 Estimated and Obtained Measures of Accuracy of Prediction Using
 Method of Largest Principal-Axes Factors

| | R_0 | R_c | $\hat{\alpha}$ | \hat{W} | W_1 | W_2 | $\hat{\psi}$ | ψ_1 | ψ_2 | |
|------------|-------|-------|----------------|-----------|-------|-------------|--------------|----------|----------|------|
| Ranks | 1 | 520 | 510 | 122 | 501 | 513 | 592 | 749 | 737 | 655 |
| | 2 | 536 | 517 | 123 | 499 | 497 | 591 | 752 | 753 | 656 |
| | 3 | 563 | 537 | 122 | 512 | 485 | 566 | 740 | 768 | 680 |
| | 4 | 604 | 573 | 117 | 544 | 482 | 586 | 707 | 783 | 660 |
| | 5 | 606 | 567 | 121 | 531 | 484 | 588 | 724 | 782 | 657 |
| | 6 | 615 | 569 | 122 | 527 | 491 | 591 | 730 | 774 | 654 |
| | 7 | 634 | 584 | 122 | 537 | 456 | 570 | 721 | 822 | 684 |
| | 8 | 635 | 576 | 126 | 522 | 450 | 568 | 740 | 830 | 687 |
| | 9 | 635 | 567 | 130 | 506 | 451 | 568 | 760 | 830 | 686 |
| | 10 | 637 | 561 | 134 | 494 | 458 | 569 | 777 | 820 | 686 |
| | 11 | 655 | 575 | 134 | 505 | 441 | 557 | 767 | 847 | 703 |
| | 12 | 661 | 574 | 136 | 499 | 432 | 566 | 777 | 863 | 695 |
| | 13 | 689 | 604 | 132 | 529 | 426 | 560 | 745 | 900 | 716 |
| | 14 | 763 | 698 | 108 | 638 | 388 | 532 | 609 | 1034 | 792 |
| | 15 | 763 | 692 | 112 | 627 | 388 | 533 | 626 | 1034 | 791 |
| | 16 | 767 | 691 | 115 | 622 | 389 | 520 | 634 | 1036 | 816 |
| | 17 | 797 | 727 | 105 | 663 | 400 | 537 | 578 | 1027 | 799 |
| | 18 | 797 | 722 | 109 | 653 | 400 | 536 | 591 | 1027 | 801 |
| | 19 | 798 | 716 | 113 | 643 | 404 | 540 | 611 | 1023 | 797 |
| | 20 | 799 | 712 | 117 | 634 | 410 | 543 | 624 | 1013 | 791 |
| | 21 | 799 | 706 | 121 | 624 | 411 | 543 | 642 | 1011 | 790 |
| | 22 | 807 | 712 | 121 | 629 | 390 | 540 | 637 | 1084 | 813 |
| | 23 | 818 | 723 | 120 | 640 | 390 | 544 | 623 | 1104 | 809 |
| | 24 | 826 | 731 | 120 | 646 | 381 | 516 | 615 | 1111 | 872 |
| | 25 | 827 | 725 | 124 | 636 | 376 | 512 | 633 | 1123 | 883 |
| | 26 | 850 | 758 | 113 | 676 | 381 | 504 | 573 | 1228 | 984 |
| | 27 | 850 | 752 | 118 | 666 | 384 | 504 | 590 | 1230 | 989 |
| | 28 | 850 | 746 | 123 | 655 | 385 | 505 | 609 | 1226 | 988 |
| | 29 | 854 | 748 | 125 | 655 | 389 | 491 | 611 | 1251 | 1030 |
| $N_0 = 75$ | | | $R_1 = 608$ | | | $R_2 = 684$ | | | | |

TABLE 5 (Cont.)
 Estimated and Obtained Measures of Accuracy of Prediction Using
 Method of Largest Principal-Axes Factors

| | R_0 | R_c | a | \hat{W} | W_1 | W_2 | $\hat{\psi}$ | ψ_1 | ψ_2 |
|----|------------|-------|-----|-------------|-------|-------|--------------|----------|----------|
| 1 | 593 | 573 | 176 | 555 | 429 | 411 | 694 | 817 | 838 |
| 2 | 593 | 552 | 191 | 514 | 429 | 411 | 742 | 817 | 837 |
| 3 | 662 | 613 | 180 | 567 | 503 | 545 | 687 | 754 | 709 |
| 4 | 681 | 617 | 187 | 560 | 492 | 537 | 701 | 766 | 718 |
| 5 | 690 | 610 | 199 | 539 | 467 | 529 | 733 | 791 | 725 |
| 6 | 722 | 634 | 199 | 557 | 426 | 479 | 718 | 836 | 779 |
| 7 | 732 | 628 | 211 | 538 | 412 | 473 | 747 | 850 | 786 |
| 8 | 744 | 626 | 222 | 526 | 424 | 477 | 770 | 846 | 790 |
| 9 | 759 | 629 | 232 | 520 | 432 | 509 | 786 | 852 | 758 |
| 10 | 803 | 683 | 215 | 581 | 347 | 413 | 712 | 984 | 909 |
| 11 | 823 | 701 | 215 | 597 | 336 | 442 | 695 | 1015 | 865 |
| 12 | 824 | 682 | 237 | 564 | 337 | 440 | 750 | 1012 | 867 |
| 13 | 830 | 671 | 256 | 542 | 317 | 427 | 789 | 1036 | 878 |
| 14 | 837 | 661 | 275 | 523 | 321 | 427 | 825 | 1045 | 887 |
| 15 | 843 | 650 | 297 | 501 | 279 | 403 | 866 | 1117 | 938 |
| 16 | 845 | 623 | 332 | 459 | 270 | 400 | 938 | 1126 | 943 |
| 17 | 848 | 592 | 372 | 413 | 264 | 419 | 1018 | 1152 | 933 |
| 18 | 853 | 566 | 411 | 375 | 257 | 387 | 1088 | 1181 | 980 |
| 19 | 872 | 589 | 418 | 398 | 305 | 447 | 1067 | 1169 | 957 |
| 20 | 906 | 682 | 364 | 513 | 222 | 411 | 892 | 1547 | 1169 |
| 21 | 911 | 660 | 409 | 478 | 196 | 391 | 959 | 1648 | 1237 |
| 22 | 915 | 625 | 473 | 426 | 198 | 376 | 1057 | 1708 | 1326 |
| 23 | 952 | 773 | 336 | 627 | 138 | 279 | 712 | 2427 | 2056 |
| 24 | 964 | 805 | 317 | 672 | 164 | 274 | 634 | 3041 | 2310 |
| 25 | 972 | 820 | 321 | 692 | 143 | 287 | 600 | 4042 | 3290 |
| 26 | 978 | 824 | 345 | 694 | 134 | 215 | 598 | 4934 | 3873 |
| 27 | 988 | 875 | 281 | 775 | 209 | 164 | 445 | 6529 | 5785 |
| 28 | 995 | 916 | 219 | 844 | 196 | 092 | 310 | 9073 | 8519 |
| 29 | 999 | 975 | 081 | 951 | -085 | -077 | 099 | * | * |
| | $N_0 = 30$ | | | $R_1 = 619$ | | | $R_2 = 672$ | | |

* Value greater than ten.

Putting (139) and (140) in (138), we obtain

$$(141) \quad E[(a'a)^2] = (N - L)(N - L + 2)\sigma^4.$$

Then, putting (141) and (137) in (136), we may write

$$(142) \quad \text{Var}(a'a) = 2(N - L)\sigma^4.$$

From (135) the variance of $\hat{\psi}$ will be

$$(143) \quad \alpha^2 = \frac{2(N + L)^2\sigma^4}{N - L}.$$

For an unbiased estimate of α^2 we use (141) and (95) to obtain

$$(144) \quad \hat{\alpha}^2 = \frac{2(1 - R_L^2)(N + L)^2}{(N - L)^2(N - L + 2)}.$$

The values for $\hat{\alpha}$ given in Table 5 were computed from the square root of (144).

In discussing Table 5, we will consider first the 16 new samples corresponding to the original-sample sizes of 120 and up. With a few exceptions, the estimated errors of prediction did not differ from the obtained values by more than one or two times the standard error of the estimate. In the full-rank case, for example, the difference between ψ and $\hat{\psi}$ was less than $\hat{\alpha}$ in eight samples, between $\hat{\alpha}$ and $2\hat{\alpha}$ in six samples, and between $2\hat{\alpha}$ and $3\hat{\alpha}$ in two samples. Ten of the obtained values fell above the estimated and six fell below. Estimates for the lower ranks tended to be more accurate. The weight-validities and their estimates evidently were less variable than the errors of prediction. Though no estimate of the standard error of \hat{W} is available, its accuracy is apparently comparable to that of $\hat{\psi}$. Taking into consideration the variability of the obtained measures of accuracy, both statistics appear to be fairly good estimates of the corresponding expected values, though their standard errors are rather larger than one could wish.

Of perhaps more significance than the absolute magnitudes of the expected values for ψ and W are the relative magnitudes from one rank to another. As a rough indication of how feasible it would be to base the choice of the rank to be used on $\hat{\psi}$, we may compare the values of ψ corresponding to the rank for which $\hat{\psi}$ was smallest with the full-rank ψ . Again considering only the 16 new samples corresponding to the original-sample size of 120 and above, we see that in 15 of the 16 instances, the reduced-rank weights so chosen gave more accurate predictions than did the full-rank weights. Some of these improvements were, of course, very small. For example, in only 8 of the 16 new samples was the reduction in total squared errors of prediction as large as 4 per cent. The largest reductions were 22.9 per cent and 21.4 per cent, both for weights from the original sample of 120 cases. Just how large the reduction would have to be to attain practical significance is, of course, debatable.

In an effort to evaluate the success of $\hat{\psi}$ as an indicator of the rank corresponding to the lowest expected error of prediction, two comparisons were made. First, it would seem reasonable to require that the total squared errors of prediction for the selected rank be closer to the lowest value obtained in a given sample than to the highest. This is the case, however, in only 9 of the 16 samples. A second comparison, intended to control for variability in the obtained errors of prediction, was made on the basis of the rank orders (from lowest to highest) of these values in the individual samples. For each member of each pair of samples corresponding to a particular original sample, the rank corresponding to rank-order 1 was determined. The rank order in the opposite member of the pair of the error of prediction corresponding to the optimal rank in the first member was then obtained. The average of these 16 rank orders was 7.4, suggesting a fair degree of stability in optimal rank. In contrast to this value, the average rank order of the errors of prediction corresponding to the selected ranks was 12.4. Since, if the ranks had been selected at random, the expected rank order would be 15, it appears that $\hat{\psi}$ does not provide a satisfactory basis for selection. However, a better basis does not appear to be available.

We consider now the results of Table 5 for the original-sample sizes of 75 and 30. For the higher ranks, both estimates appear to break down completely. For the lower ranks, taking into account the large standard errors, the two estimates appear to do about as well as in the larger samples. Because of these large standard errors, however, $\hat{\psi}$ and \hat{W} are not very helpful as guides to the absolute magnitude of the corresponding expected values. If taken as an aid to judgment rather than as an index to be applied blindly, $\hat{\psi}$ in particular might be of value in arriving at an optimal rank. In the original sample of size 30, the lowest value of $\hat{\psi}$ for ranks below 24 occurred for rank 3. Very little judgment is required to select a rank-3 solution in preference to a solution of rank 24 or more on a sample of 30 cases. As it turned out, the optimal rank was in fact 3 in both cross-validation samples. In the original sample of size 75, the alternative to a rank-4 solution would be one of rank 14 or more. For samples of 75 cases an optimal rank of 14 is certainly possible, though unlikely. In any event, it appears that, providing unrealistically low values for higher ranks are ignored, $\hat{\psi}$ is potentially of some value in deciding what rank to use for small samples as well as for large ones.

It will be recalled that in deriving $\hat{\psi}$ and \hat{W} , the assumption was made that the factor loadings of the predictor matrix would be constant from sample to sample. Thus the very limited success of these statistics may be due to the failure to take sampling variation of the factor loadings into account. This, of course, could not have been done within the context of regression theory, since there only the criterion variable is considered random. The regression model was selected for this study largely on the basis of its simplicity, but also on the grounds that it is the model generally used in con-

nection with prediction problems. However, it seems likely that an analysis of prediction problems in terms of the multivariate normal model of correlation theory or in terms of some other model where the predictor variables are considered random would lead to more successful estimates of accuracy of prediction than those obtained using regression theory.

CHAPTER 4

SUMMARY AND CONCLUSIONS

The primary concern of this study has been with the possibility of using reduced-rank solutions for regression weights to increase the accuracy of prediction obtainable in future samples. Using regression theory, a general factor model for reduced-rank prediction was developed. It was shown that, if errors in the criterion observations are not to be capitalized upon, the optimal basis for determining a lower-rank solution will be the amount of variance accounted for in the predictor data matrix. Thus the best alternative to reduced-rank methods that seek to obtain the maximum multiple correlation with the criterion would be the method of largest principal-axes factors, as suggested by Horst (1941). Estimates of the weight-validities and total squared errors of prediction to be expected when a particular set of weights is applied in future samples were also derived.

An empirical comparison of five particular reduced-rank methods was carried out, using 29 predictors and with partial replication on five criteria. Weights were computed on samples ranging from 30 to 435 cases. As expected, the method of largest principal-axes factors was markedly superior to the other methods tested. This superiority was quite general, appearing in all samples for some criteria, and in some samples for all criteria. The above finding, together with the very poor showing of the method of smallest principal-axes factors, supports the conclusion regarding the importance of predictor variance accounted for by the lower-rank system. The fact that the largest principal-axes factors tended to give more accurate predictions than did the principal-axes factors having the highest multiple correlation with the criterion suggests the desirability of selecting predictors independently of the criterion observations. The exceptions to this trend for the larger original-sample sizes on some criteria indicates the desirability of developing some sort of statistical test for deciding when the predictor-selection methods using the criterion observations may be advantageously applied.

Although their standard errors were rather large, especially in small samples, the estimates of weight-validity and of total squared errors of prediction to be expected in future samples appeared to be reasonably serviceable as regards absolute magnitude. As to relative magnitude from one rank to another, however, it may be questioned whether a rank chosen on the basis of these estimates would be preferable to a rank chosen at random. As estimates of either absolute or relative magnitude, it seems likely that the

statistics derived here could be substantially improved upon if variation in the predictor variables or in their factor loadings were taken into account. Without such improved estimates, the large potential advantages of reduced-rank methods demonstrated here cannot be fully realized. Thus it would seem well worthwhile to undertake an analysis of prediction problems using a statistical model which, unlike regression theory, treats the predictors as random variables.

Until more efficient methods are developed, it is suggested that a regression equation based on the subset of largest principal-axes factors for which $\hat{\psi}$ is smallest will be the best available. For samples with less than, say, 50 degrees of freedom, this procedure must be supplemented by a subjective process to the extent of ignoring low values of $\hat{\psi}$ for ranks of say, ten or more. Although this procedure leaves considerable room for improvement, the relevant evidence seems sufficiently favorable to warrant further empirical research. At any rate, the strong possibility has been raised that the conventional full-rank weights can almost always be improved upon even in samples of several hundred cases. Such weights, moreover, may give predictions only slightly more accurate than those made from weights obtainable with samples of as few as 30 cases.

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